Combinatorics, geometry, and number theory problems

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1 Introduction

In this document we have collected problems from combinatorics, combinatorial number theory, computational number theory, and geometry that are hopefully, engaging and challenging for High School students, and do-able by them.

Our main aim in collecting and presenting these problems is to provide High School students with a fun collection: problems that stimulate and challenge their mathematical thinking, and are fun to work on.

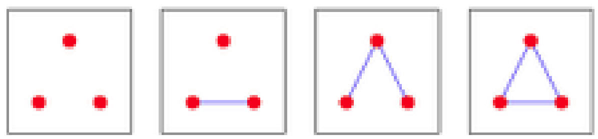
1.1 Several flavors of problems

These problems tend to come in several distinct "flavors". One flavor consists of curious, often tantalizing, yet somewhat contrived, puzzles. These are often engaging, fun to work on, and sometimes offer beautiful insights. Working on these problems can be very satisfying and great fun.

Another flavor consists of *investigations*, in which there is not a single clearly stated problem, but instead a proposed investigation of some curious phenomenon.

Thirdly, another, generally very hard, flavor of problems consists of *unsolved problems*. These are mostly problems that several, if not many, professional mathematicians have thought about long and hard and have not yet been able to resolve. While it's sensible to be aware of these unsolved problems, and perhaps to think for a short time what the problem entails, it is generally not a wise move to spend a long time thinking about these hard unsolved problems simply because the chance of solving them is very low, and time spent on them means time taken away from other potentially solvable problems.

Another flavor, which we will into go into in detail here consists of problems that come from other branches of mathematics. For example, in graph theory, a fundamental problem is how many essentially different graphs there are with a given number of vertices. For example, here are the 4 essentially different graphs with 3 vertices:



Counting, or "enumerating", the number of essentially different graphs with n vertices is a fundamental problem in graph theory and utilizes a method known as Polya-Burnside enumeration.

1.2 Software for computation

Occasionally you might need, or want to, use mathematical software to carry out computations. Two open source software packages for this purpose are:

1. SageMath
2. Geogebra

1.3 New delights and unexpected connections

As you peruse these problems and investigations bear in mind the words of the late, great Bill Thurston,

Figure 1: William P. Thurston

who wrote:

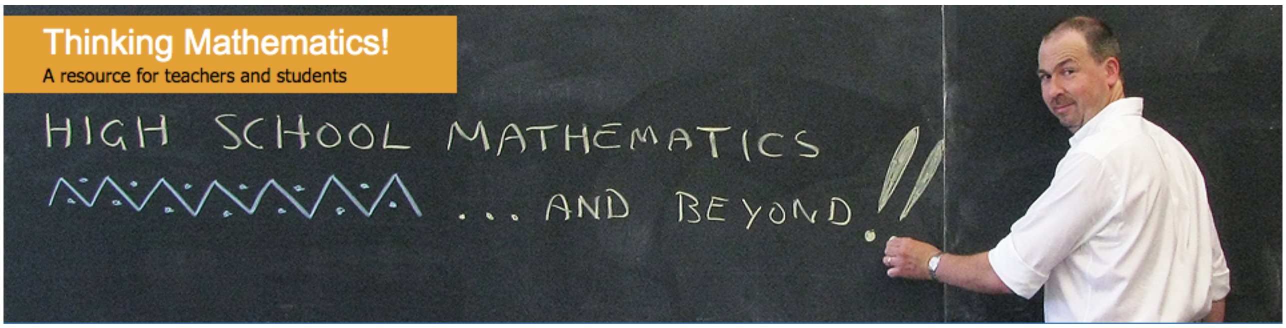
> "Mathematics has a remarkable beauty, power, and coherence, more than we could have ever expected. It is always changing, as we turn new corners and discover new delights and unexpected connections with old familiar grounds."

As you engage with a problem, and potentially get stuck and confused, bear in mind some other helpful words from Bill Thurston:

“Mathematics is a process of staring hard enough with enough perseverance at the fog of muddle and confusion to eventually break through to improved clarity. I’m happy when I can admit, at least to myself, that my thinking is muddled, and I try to overcome the embarrassment that I might reveal ignorance or confusion. Over the years, this has helped me develop clarity in some things, but I remain muddled in many others. I enjoy questions that seem honest, even when they admit or reveal confusion, in preference to questions that appear designed to project sophistication.”

1.4 Acknowledgements

A major stimulus and source of inspiration for us in compiling this list of mathematical problems for High School students and undergraduates, has been Dr. James Tanton (On Twitter: @jamestanton)



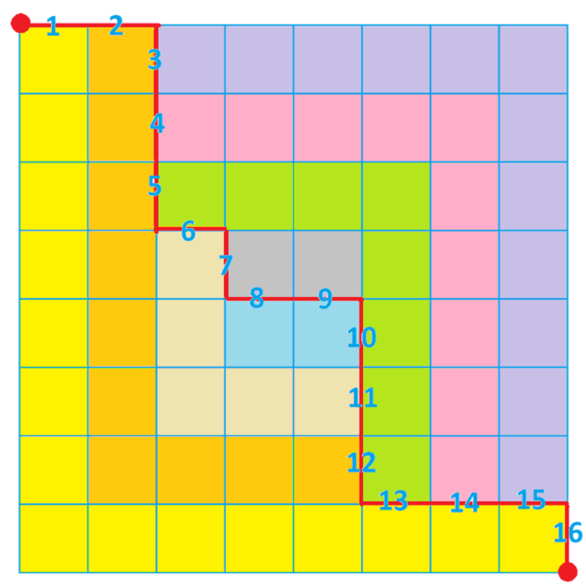
and you will see numerous acknowledgements to James on individual problems, where he contributed either the whole problem itself, or the stimulus for us to enlarge an original idea of his.

2 Curious, tantalizing puzzles

2.1 Coloring a square grid

This problem is from Klee, S., Malkin, K. & Pevtsova, J. (2021). Math Out Loud: An Oral Olympiad Handbook (Vol. 27). American Mathematical Society, and was posted on Twitter by James Tanton (@jamestanton):

Draw a path of horizontal and vertical steps from the top left corner to the bottom right corner of an $n \times n$ grid. If step 1 is horizontal, color all cells below it, turn 90 degrees, color rightward all squares back to the path. If step 1 is vertical, go rightward, then down. Do this for the first n steps. Is the whole grid sure to be colored?



2.2 Which integer sequences are possible?

(1) Sequences $a_0, a_1, a_2, a_3, \ldots$ of non-negative integers a_n are constructed subject to the following constraints:

- $a_0 = 0$
- $a_1 = 1$
- for all $n \geq 1, a_n = \lfloor \frac{a_{n-1}+a_{n+1}}{2} \rfloor$, where for a real number x, $\lfloor x \rfloor$ is the *floor* of x: the greatest integer less than, or equal to, x.

What are the possible integer sequences that satisfy these constraints?

Thanks to James Tanton (@jamestanton on Twitter) for this idea.

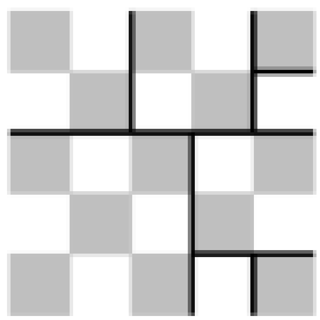
(2) As a variant, investigate and characterize positive integer sequences $a_1, a_2, a_3, \ldots$ constructed subject to the following constraints:

- $a_1 = 1$
- $a_2 = 1$
- for all $n \geq 1, a_n = \lfloor \sqrt{a_{n-1} * a_{n+1}} \rfloor$, where, as before, for a real number x, $\lfloor x \rfloor$ is the *floor* of x: the greatest integer less than, or equal to, x.

2.3 Cutting a checker board

This problem is adapted from Klee, S., Malkin, K. & Pevtsova, J. (2021). Math Out Loud: An Oral Olympiad Handbook (Vol. 27). American Mathematical Society:

A 5×5 checkerboard is cut along the grid-lines into a number of smaller square boards. Prove that the total length of the cuts is a multiple of 4.



What about a 25×25 checkerboard?

For which n is this true for an $n \times n$ checkerboard?

Is this true, for example, for n even?

2.4 Can Robert stop Kayley?

This problem is adapted from Klee, S., Malkin, K. & Pevtsova, J. (2021). Math Out Loud: An Oral Olympiad Handbook (Vol. 27). American Mathematical Society:

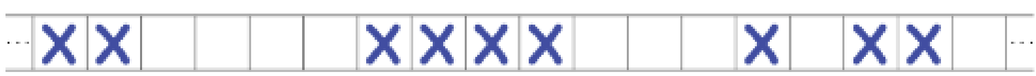
Two players, Kayley and Robert take turns in making and erasing "X" marks on an infinite tape of squares, as follows:

1. Kayley goes first, and at each turn marks 2 of the squares of the tape (not necessarily adjacent) with an "X"

2. Robert takes turns after Kayley and can erase any block of adjacent X's

What length blocks of adjacent X's can Kayley make without being stopped by Robert?

In the picture below there are blocks of length 1, 2 and 4 of adjacent X's:

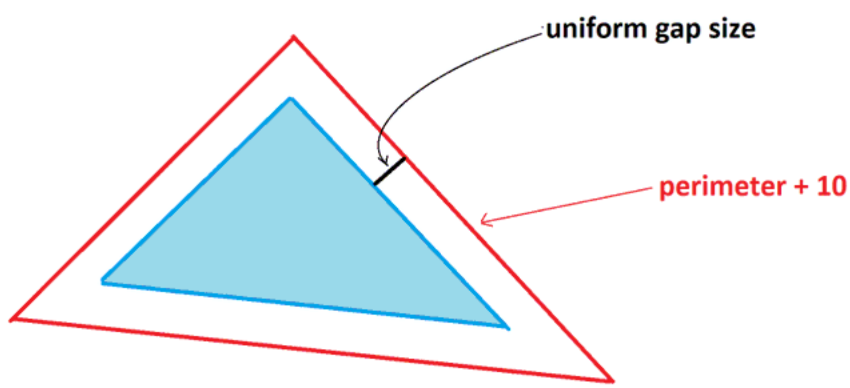


2.5 Wrapping a string around a polygon

A string, 10 units longer than the perimeter of a polygon, is wrapped around the polygon to make a similar polygon with a gap of uniform size between the two perimeters.

In terms of the area and the perimeter of the original polygon, how big is the gap?

Here is a picture for the case of a triangle:



Does it make difference if the polygon is, or is not, convex?

Thanks to James Tanton for this problem.

2.6 Tiling with tetrominos

This problem is from James Propp.

Figure 2: James Propp

An Aztec diamond of order n is

> "a region composed of $2n(n+1)$ unit squares, arranged as a stack of $2n$ centered rows of squares, with the k^{th} row having length $\min(2k, 4n-2k+2)$"

Jockusch, W., Propp, J., & Shor, P. (1998). Random domino tilings and the arctic circle theorem

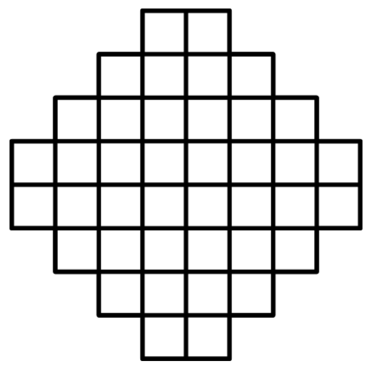


Figure 3: Aztec diamond of order 4

We are going to try to tile Aztec diamonds with tetrominos.

The 5 different tetrominos are:

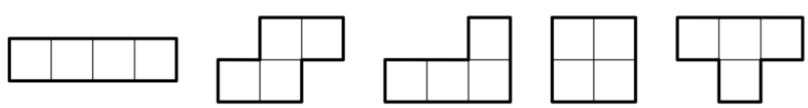


Figure 4: The 5 distinct tetrominos

Here is a tiling of the Aztec diamond of order 3 by tetrominos:

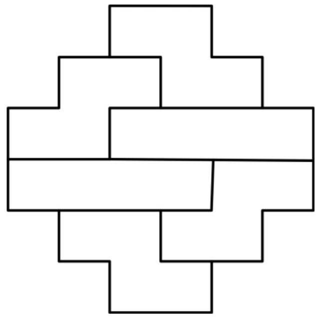
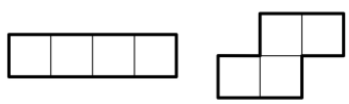
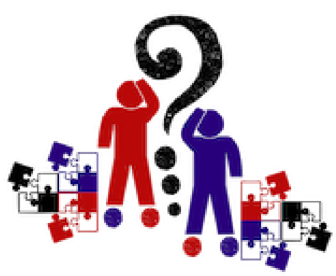


Figure 5: Aztec diamond of order 3 tiled by tetrominos

For which values of n can an Aztec diamond of order n be tiled by the first two tetrominos to the left of Figure 4 (the tetrominos shown below)?



Another tiling problem: is it possible to tile a *rectangle* using all five tetrominos?



2.7 How many sixes from throwing a die many times?

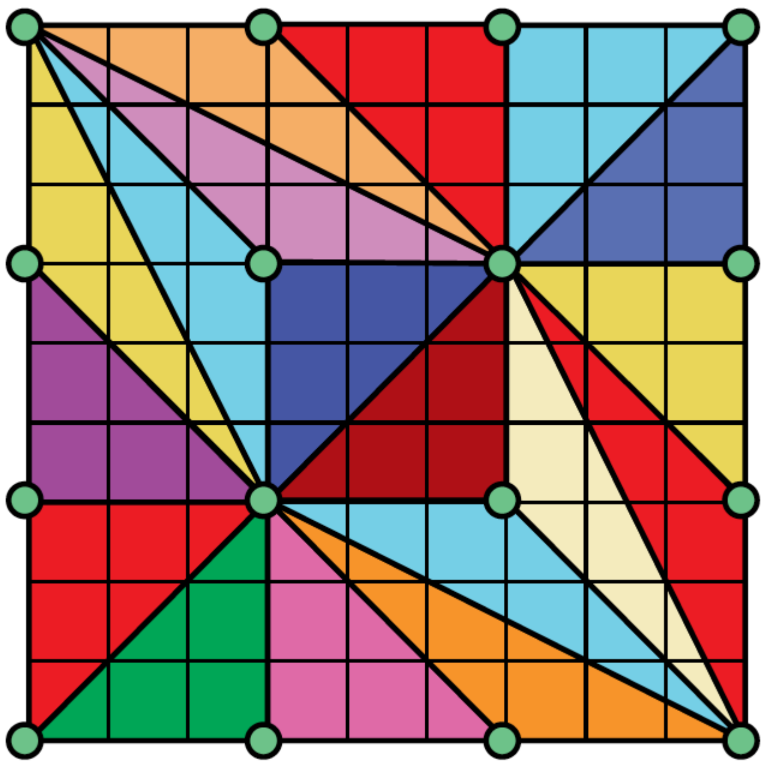
Imagine we throw a n dice all at once, and record how many 6s we see.

As a function of n,

- What is the probability of getting an even number of 6's?
- What is the probability of getting an odd number of 6's?
- What is the probability of getting a number of 6's that leaves a remainder of 0 (mod 3)?
- What is the probability of getting a number of 6's that leaves a remainder of 1 (mod 3)?
- What is the probability of getting a number of 6's that leaves a remainder of 2 (mod 3)?

2.8 Subdividing a square into triangles

It is possible to divide a nine-by-nine grid of squares into 18 triangles of equal area, each with a vertex at an intersection point on the grid:



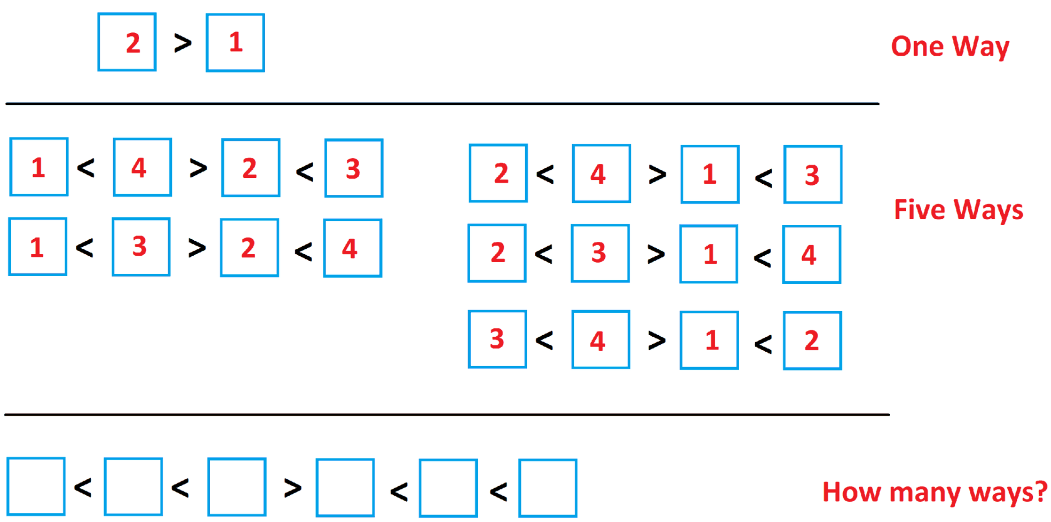
Is it possible to divide the nine-by-nine grid of squares into an odd number of triangles of equal area (with vertices at grid points)?

This problem is from James Tanton (@jamestanton on Twitter).

What about $n \times n$ grids? Does it make a difference if n is odd or even?

2.9 Arranging the numbers 1, 2, ... , 2n-1, 2n

In how many ways can one arrange the numbers 1, 2, 3, 4, 5, 6 and respect the inequalities between adjacent terms as shown? (All inequalities are $<$ except for the middle one.)

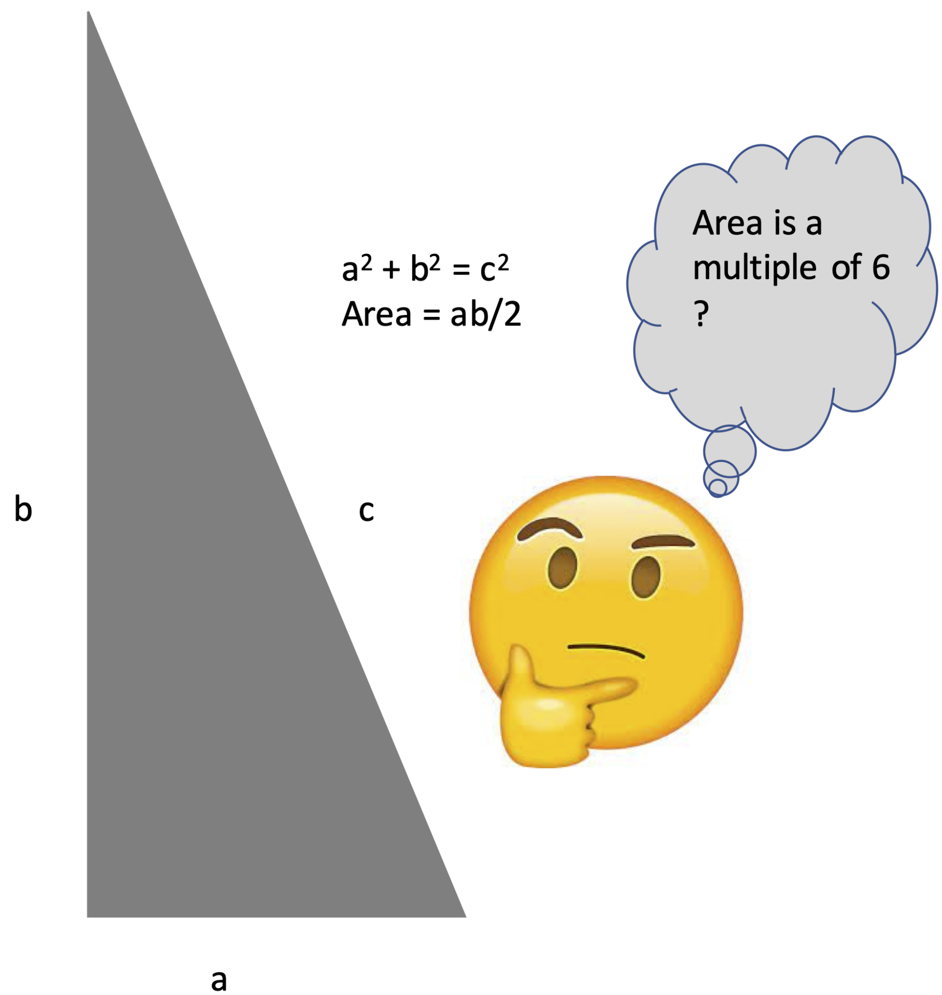


How about 1, 2, 3, 4, 5, 6, 7, 8? Or 1, 2, 3, 4, 5, 6, 7, 8, 9, 10?

Thanks to James Tanton for this problem.

2.10 Area of integer right triangles

Can you prove conclusively that the area of any right triangle with integer side-lengths is sure to be a multiple of 6?



Thanks to James Tanton (@jamestanton on Twitter) for this problem.

2.11 Pythagorean triples

If (a, b, c) is a Pythagorean triple - meaning a, b, c are positive integers and $a^2 + b^2 = c^2$ - can you prove conclusively that:

- at least one of a or b is sure to be divisible by 3
- at least one of a or b is sure to be divisible by 4, and
- at least one of a, b, or c is sure to be divisible by 5

?

$$a^2 + b^2 = c^2$$

(3,4,5) (6,8,10) (7,24,25)

(5,12,13) (20,21,29) (8,15,17)

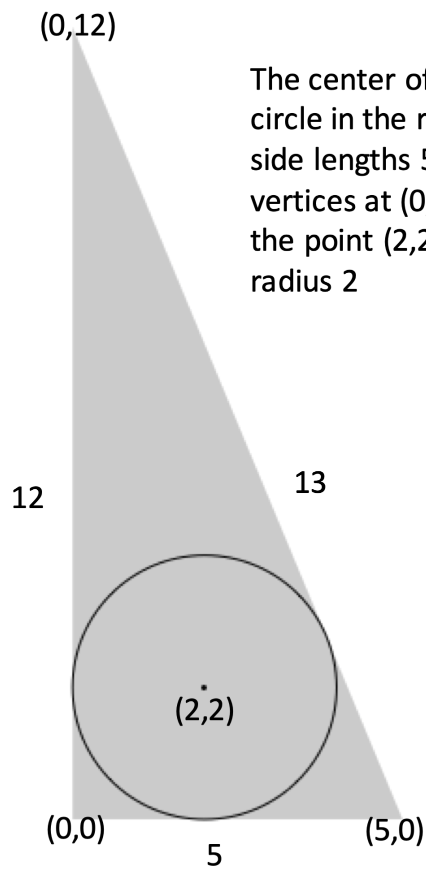
(20,99,101) (48,55,73) (17,144,145)

Pythagorean Triples

Thanks to James Tanton (@jamestanton on Twitter) for this problem.

2.12 Inscribed circles in integer right triangles

In the right triangle below, with side lengths 5, 12, 13, the largest inscribed circle has integer radius:

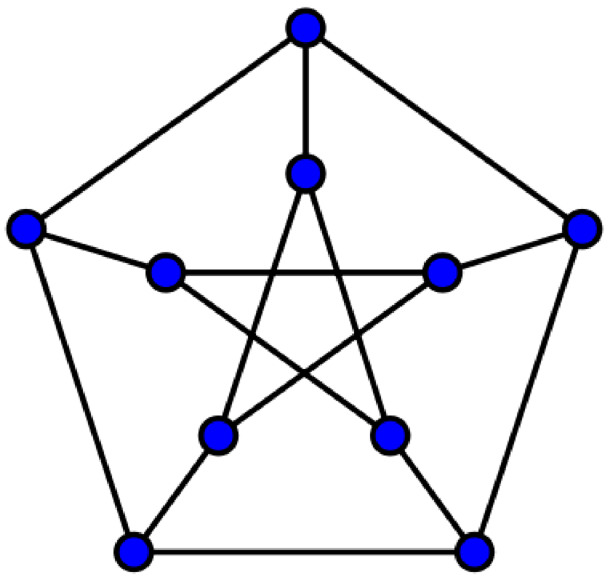


Can you prove conclusively that the radius of the largest circle one can draw inside any right triangle with integer side-lengths is sure to have an integer radius?

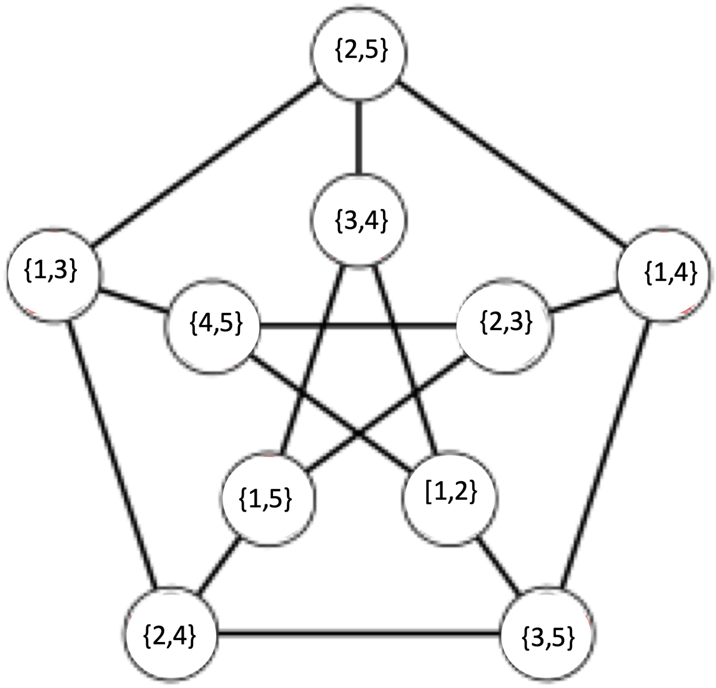
Thanks to James Tanton (@jamestanton on Twitter) for this problem.

2.13 Symmetries of the Petersen graph

The Petersen graph is the following graph with 10 vertices and 15 edges:



We can label the vertices of the Petersen graph with pairs of integers chosen from $\{1, 2, 3, 4, 5\}$ so that two vertices are joined by an edge if and only if their labels are disjoint (that is, have no numbers in common):



A permutation of the set $\{1, 2, 3, 4, 5\}$ is a rearrangement of that set. There are $5! = 120$ permutations of $\{1, 2, 3, 4, 5\}$.

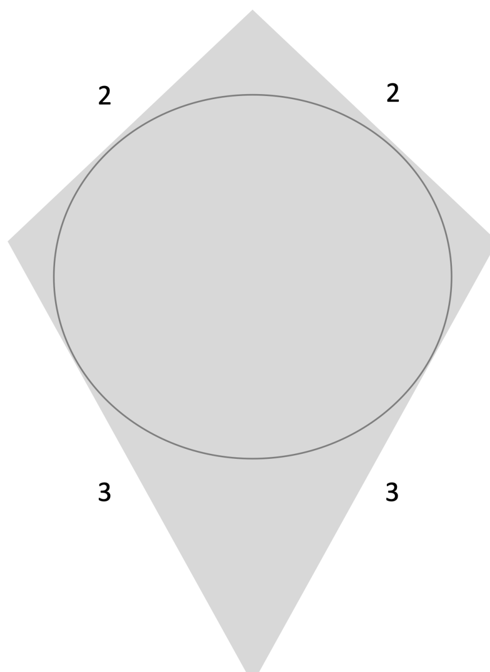
Show that every permutation of $\{1, 2, 3, 4, 5\}$ gives a symmetry of the Petersen graph: each permutation maps the labels of a vertex to a new label so that two vertices are joined by an edge if and only if the permuted labels are joined by an edge.

Can you show there are no other symmetries of the Petersen graph?

2.14 Quadrilaterals with an inscribed circle

A convex quadrilateral Q is constructed such that:

- Each of the 4 side lengths of Q is one of the integers $1, 2, 3$. Note: repetitions are allowed so some, or all, of the sides will be of equal length.
- Q contains a circle tangent to each of the 4 sides.



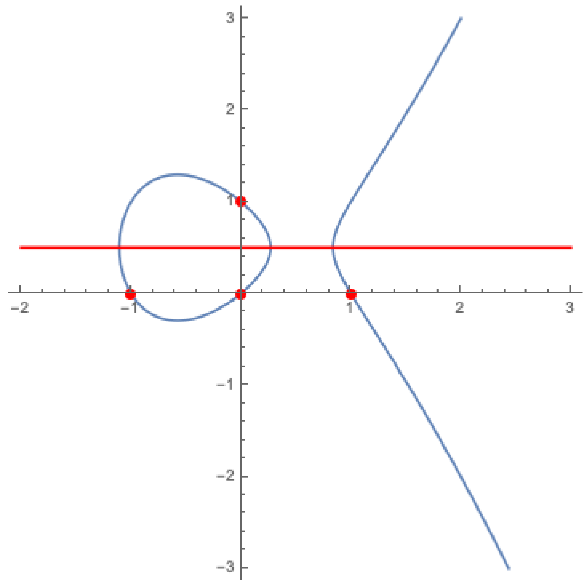
How many such quadrilaterals are there, and what are their 4 side lengths L_1, L_2, L_3, L_4?

What are the centers and radii of their inscribed circles?

What are the areas of the quadrilaterals?

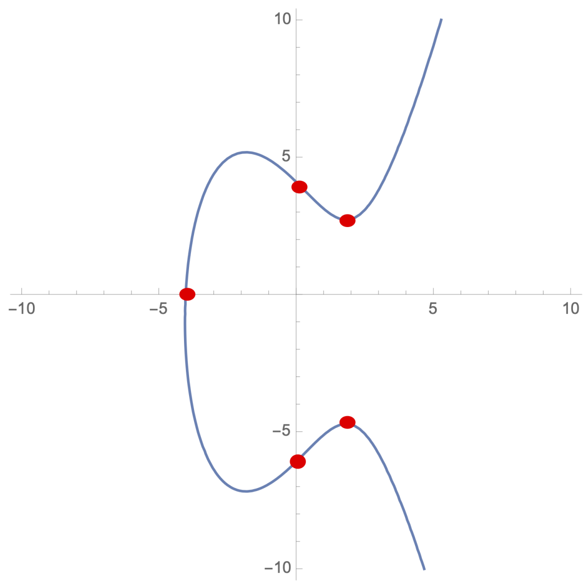
2.15 Features of elliptic curves

The set of points $E := \{(x, y) \in \mathbb{R}^2 : y^2 - y = x^3 - x\}$ is an example of an elliptic curve:



Find the red points and the red line - about which the curve is symmetric (why is the curve symmetric about that line?).

The elliptic curve $y^2 + 2y = x^3 - 10x + 25$ is shown below - find the indicated points on the curve:



2.16 How many increasing trees?

An "*increasing tree*" is a tree with n vertices, labelled 1, 2, ..., n, with the root of the tree labelled "1", such that the vertex labels are *increasing* as we travel down the tree from the root:

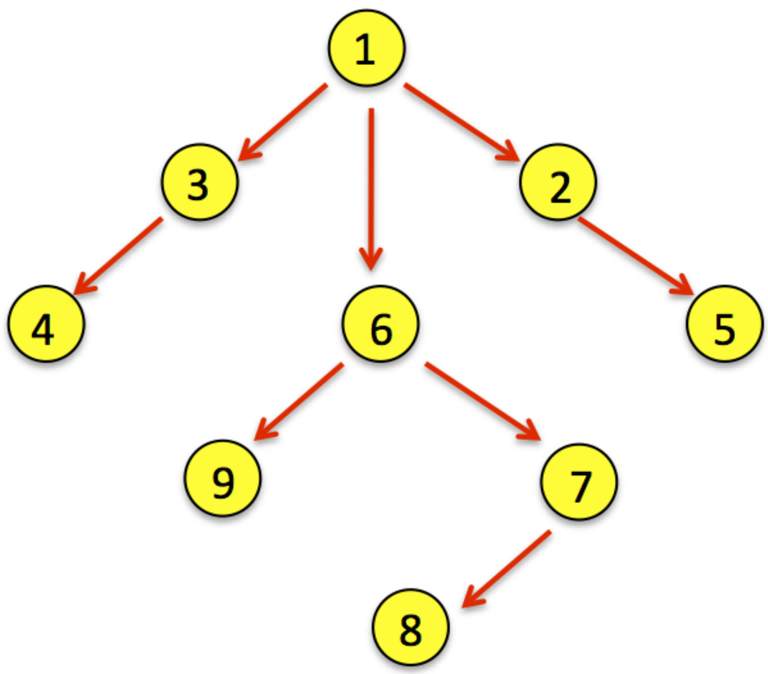


Figure 6: An increasing tree on 9 vertices

For each natural number n, how many increasing trees with n vertices are there?

Thanks to Per Alexandersson for this problem.

2.17 The Calkin-Wilf tree

The Calkin-Wilf tree is a binary tree with root $\frac{1}{1}$ and each entry $\frac{a}{b}$ branches into a l*eft child* $\frac{a}{a+b}$ and *right child* $\frac{a+b}{b}$:

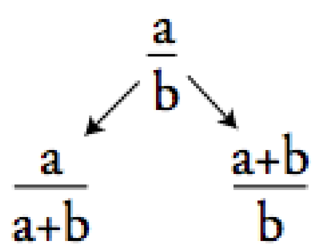


Figure 7: $\frac{a}{b}$ has left child $\frac{a}{a+b}$ and right child $\frac{a+b}{b}$

The rational numbers $\frac{a}{b}$ in the Calkin-Wilf tree occur in *levels*, where level 0 is $\{\frac{1}{1}\}$ and level n, for $n \geq 1$, consits of the left and right children of rational numbers in level $n-1$:

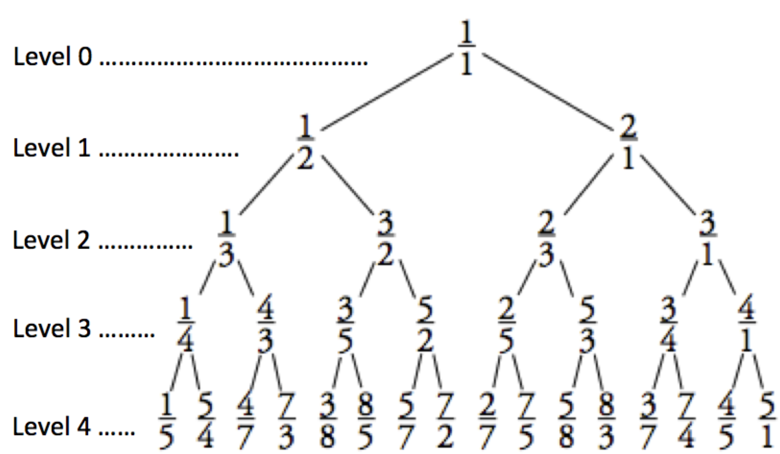


Figure 8: Levels 0 through 4 of the Calkin-Wilf tree

Can you prove the following about the Calkin-Wilf tree?

- For each entry $\frac{a}{b}$ in the Calkin-Wilf tree, $a \geq 1, b \geq 1$ and the greatest common divisor (GCD) of a and b is 1.
- Every rational number $\frac{a}{b}$ with $a \geq 1, b \geq 1$ and $\text{GCD}(a, b) = 1$ occurs once and only once in the Calkin-Wilf tree.
- The list, left to right, of denominators in level n of the Calkin-Wilf tree is the reverse of the numerators, left to right, in level n.
- There are 2^n terms in level n.
- The sum of the numerators (= the sum of the denominators) in level n is 3^n.

Investigate a formula for the *average* value of all terms in level n of the Calkin-Wilf tree.

2.18 Average of triangular numbers

The n^{th} triangular number is $T(n) = \frac{n\times(n+1)}{2}$.

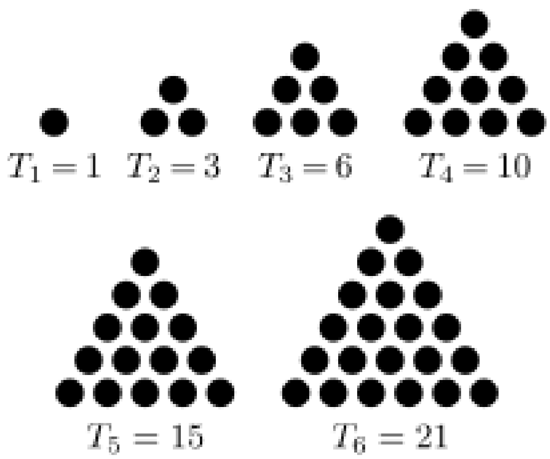


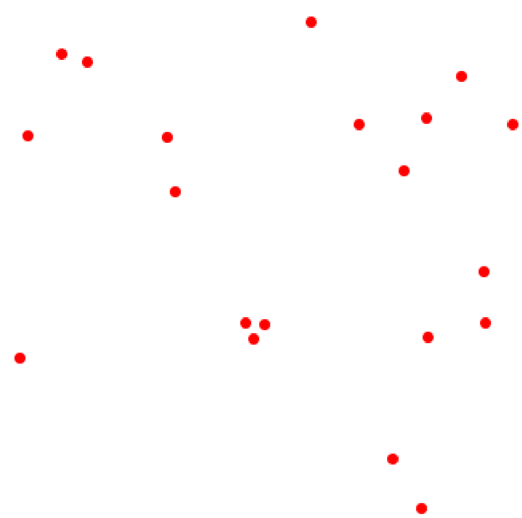
Figure 9: The first six triangular numbers

For which n is the average, $\frac{1}{n}\sum_{k=1}^{n} T(k)$, of the first n triangular numbers $T(1), T(2), \ldots, T(n)$ an integer?

3 Investigations

3.1 An algorithm for a printing path

Suppose we have a collection of points in a plane, for example (but the points might also be in 3-dimensional space):

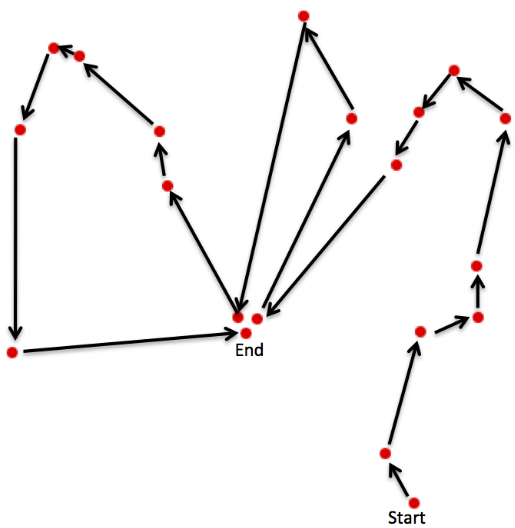


A *printing path* on the collection of points consists of:

1. a point designated as "start"
2. a point designated as "end"
3. for each point other than the start and end, an arrow into that point from another point, and an arrow out of that point to another point
4. a arrow out of the start point to another point
5. an arrow into the end point from another point

6. the arrows do not cross or meet except at the specified points

In the language of directed graphs, a *printing path* on the points is a plane directed graph with the points as vertices in which each vertex has in-degree 1 and out-degree 1, except for the start point which has in-degree 0 and out-degree 1, and the end point which has in-degree 1 and out-degree 0.



Can you devise an algorithm that, given the points, constructs a printing path on the points? Is this always possible? Can you devise an algorithm so that the total length of the arrows joining the points is as small as possible?

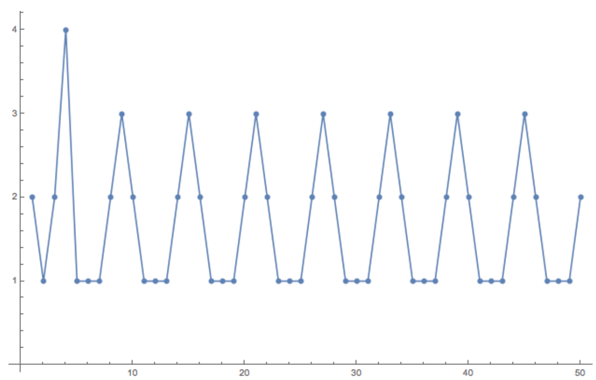
Thanks to Alfa Heryudono for this problem.

3.2 Investigating Mahler's 3/2 problem

Kurt Mahler was a German born mathematician who, among other appointments, spent many years as Professor of Mathematics at the Institute for Advanced Studies, the Australian National University. Mahler conjectured that for every real number x there is a positive integer n such that the fractional part of $x(\frac{3}{2})^n$ is $\frac{1}{2}$ or greater. This problem is still unsolved.

Investigate for a specific real number - for example $x = \frac{1+\sqrt{5}}{2}$ - and for each natural number n, what is the least positive integer k_n for which the fractional part of $x^n(\frac{3}{2})^{k_n} \geq \frac{1}{2}$. In other words, we are assuming that for *each* of $x, x^2, x^3, x^4, \ldots$ there will be a positive integer k_n for which the fractional part of $x^n(\frac{3}{2})^{k_n} \geq \frac{1}{2}$, and we want to compute the first such k_n given a power x^n of x.

Is there some pattern? Can you make sense of, or explain, any patterns you see? For example, for $x = \frac{1+\sqrt{5}}{2}$, below is a plot for each n (horizontal axis) of the first k_n (vertical axis) for which the fractional part of $x^n(\frac{3}{2})^{k_n} \geq \frac{1}{2}$:



3.3 Number of primes between successive Fibonacci numbers

The n^{th} Fibonacci number, denoted $F(n)$ is defined recursively as:

- $F(0) = 0$
- $F(1) = 1$
- $F(n) = F(n-1) + F(n-2)$ for $n \geq 2$

Investigate how the number of primes between $F(n)$ and $F(n+1)$, inclusive, grows with n.

As a variant on this problem, Investigate how the number of primes between the n^{th} and $(n+1)^{st}$, inclusive, Catalan numbers grows with n.

3.4 A prime coincidence?

Ciara had learned from her mathematics teacher about modular arithmetic, and was especially intrigued by arithmetic in the set $\mathbb{Z}_p = \{0, 1, 2, \ldots, p-1\}$ for p a prime number.

Ciara was fascinated that it was possible to do division by non-zero elements of $\mathbb{Z}_p$ because, thanks to the fact that p is prime, for every $0 \neq x \in \mathbb{Z}_p$ there is an "inverse" $0 \neq y \in \mathbb{Z}_p$ for which $xy \equiv 1(\text{mod p})$.

Ciara wrote a computer program that, given a prime number p, would print out the inverse of each non-zero element of $\mathbb{Z}_p$.

x	inverse of x, mod 13
1	1
2	7
3	9
4	10
5	8
6	11
7	2
8	5
9	3
10	4
11	6
12	12

Being in a playful mood, Ciara calculated the difference between x and inverse of x, mod p for each $x \in \mathbb{Z}_p$ and then formed the sum of all these numbers:

x	inverse of x, mod 13	x-inverse of x, mod 13
1	1	0
2	7	8
3	9	7
4	10	7
5	8	10
6	11	8
7	2	5
8	5	3
9	3	6
10	4	6
11	6	5
12	12	0

The total of all $x - inv(x)$ mod 13 is 65.

Ciara did these calculations for the first 25 odd primes p (primes other than 2) and found that the total was always $\frac{p(p-3)}{2}$:

odd prime p	Ciara's total for p	p(p-3)/2
3	0	0
5	5	5
7	14	14
11	44	44
13	65	65
17	119	119
19	152	152
23	230	230
29	377	377
31	434	434
37	629	629
41	779	779
43	860	860
47	1034	1034
53	1325	1325
59	1652	1652
61	1769	1769
67	2144	2144
71	2414	2414
73	2555	2555
79	3002	3002
83	3320	3320
89	3827	3827
97	4559	4559

Could this be just a coincidence?

If not, why might it be true?

3.5 Numbers with a factor having the same number of 1s, base 2

James Tanton (@jamestanton) asked the following question on Twitter, Wednesday, August 8, 2018:

"Which positive integers n have a factor $k < n$ so that n and k have the same number of 1s in binary?"

Is it obvious this is true for powers of 2?

What about prime numbers n?

And what about squares of primes?

As a variant, which positive integers n have a factor $k < n$ so that n and k have the same number of 0s in binary?

3.6 Occurrences of the digit 2 in the base 3 expansion of powers of 2

The base 3 expansion of $2^8 = 256$ is 100111, because

$$1 \times 3^5 + 0 \times 3^4 + 0 \times 3^3 + 1 \times 3^2 + 1 \times 3^1 + 1 \times 3^0 = 256$$

There is no digit "2" in the base 3 expansion of $2^8 = 256$, and a famous conjecture of Paul Erdös is that this is the last n for which 2^n has no digits 2 in its base 3 expansion: namely, Erdös conjectures that for all $n > 8$, the base 3 expansion of 2^n does contain the digit 2 at least once.

Can we be more quantitative about this? What does experiment suggest is the *average number* of occurrences of the digit 2 in the base 2 expansion of 2^n?

In other words, suppose we compute the number of occurrences of the digit 2 in the base 3 expansion of 2^k for all $k \leq n$ and form the average of all those numbers. What is a good estimate of how that average varies with n?

3.7 Binary disjoints

For a positive integer n, call a positive integer $k < n$ a *binary disjoint* of n if the binary representations of k and n have no 1's in common places.

For example, 5 is a binary disjoint of 10 because $5 = 101$ base 2, and $10 = 1010$ base 2: 5 has 1's in the 2^0 and 2^2 places, while 10 has 1's in the 2^1 and 2^3 places.

10 is the sum of its binary disjoints: 1, 4, 5. Is there any other positive integer that is the sum of its binary disjoints?

James Tanton (@jamestanton) asked on Twitter on April 14, 2018, which n are a multiple of each of their binary disjoints - are these just the numbers of the form $n = 2^k - 2$, or are there other such numbers?

Which n have only 1 as a binary disjoint?

Note that 9 and 25 have the same set of binary disjoints: 2, 4, 6. Let's call such a pair "binary disjoint friends". What other pairs of binary disjoint friends can you find?

3.8 Probability random points are in convex position

Suppose that n points are chosen uniformly randomly and independently from inside the square $[0, 1] \times [0, 1]$.

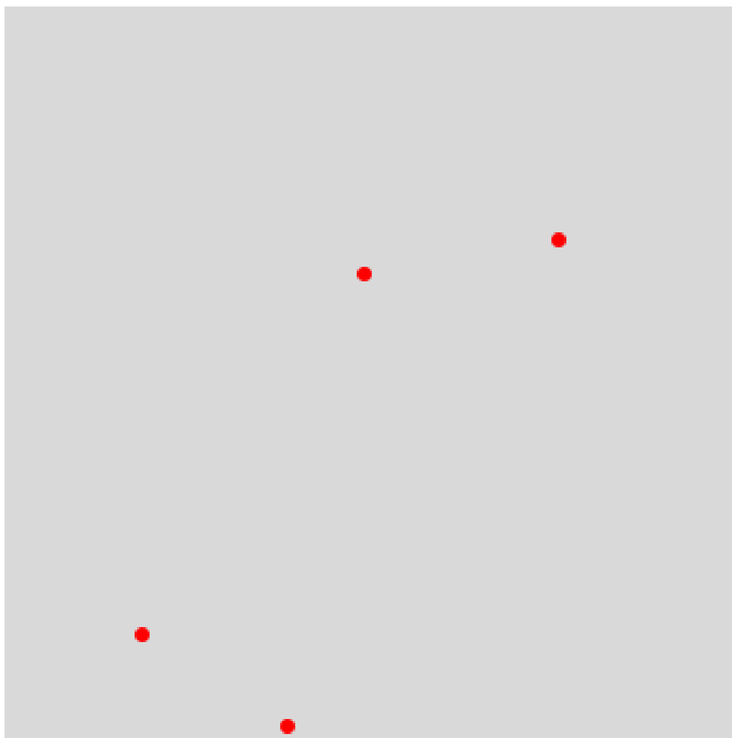


Figure 10: 4 uniformly random points in a square

The points are in *convex position* if each point is an extreme point of the convex hull of all the points.

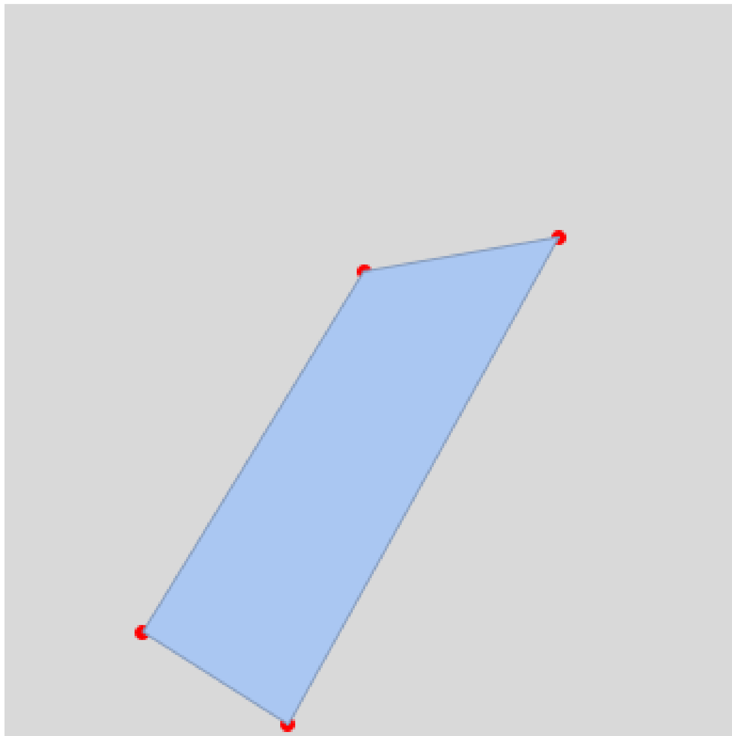


Figure 11: Convex hull of 4 uniformly random points in a square

What is the probability, as a function of n, that n uniformly random points in the square are in convex position ?

3.9 Numerators of the fractional parts of powers of 3/2

It is a famous long-standing problem whether the fractional parts of $(\frac{3}{2})^n$ are uniformly distributed in the interval $[0, 1]$.

If we denote by $\mathrm{fp}(x)$ the fractional part of a real number x, "uniformly distributed" means for all $0 \le a < b \le 1$

$$\frac{\#\{k \le n : a \le \mathrm{fp}((3/2)^k) \le b\}}{n} \to b - a \text{ as } n \to \infty$$

This is a notoriously difficult problem on which mathematicians are actively working.

What, however can you say about the behavior, or even the *average behavior*, of the numerators of the fractional parts of $(\frac{3}{2})^n$?

The first 20 numerators are:

1, 1, 3, 1, 19, 25, 11, 161, 227, 681, 1019, 3057, 5075, 15225, 29291, 55105, 34243, 233801, 439259, 269201

4 Unsolved problems

Remember that as challenging and intriguing as you may find these problems they are mostly problems that several, if not many, professional mathematicians have thought about long and hard and have not yet been able to resolve. While it's sensible to be aware of these unsolved problems, and perhaps to think for a short time what the problem entails, it is generally not a wise move to spend a long time thinking about these hard unsolved problems simply because the chance of solving them is very low, and time spent on them means time taken away from other potentially solvable problems.

4.1 How often does each positive integer occur in Pascal's triangle?

The number 1 clearly occurs infinitely often in Pascal's triangle. However integers $n > 1$ occur only a finite number of times.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

Can you devise ways of calculating for any given integer $n > 1$ how often n occurs in Pascal's triangle?

A famous conjecture of David Singmaster is that there is a number N such that every integer $n > 1$ occurs no more than N times in Pascal's triangle.

So far no one has found an $n > 1$ that occurs more than 8 times in Pascal's triangle (so maybe $N = 8$?).

4.2 A prime between successive powers of an integer ?

Samantha had heard about a famous unsolved problem: that there is always a prime number between n^2 and $(n+1)^2$, for all natural numbers n.

Being a quantitative data-oriented person, Samantha did some calculations and came up with a stronger thought: "It seems to me", said Samantha, "on the basis of calculational evidence, that the number of primes between n^2 and $(n+1)^2$ is always greater than $\frac{n}{9}$".

Could Samantha be right?

What does experiment suggest is the *average number* of primes between n^2 and $(n+1)^2$? In other words, suppose we compute the number of primes between k^2 and $(k+1)^2$? for all $k \leq n$ and form the average of all those numbers. What is a good estimate of how that average varies with n?

4.3 Is there a polyomino or order 5?

A polyomino is a connected collection of squares each of which is connected to another square along an entire edge:

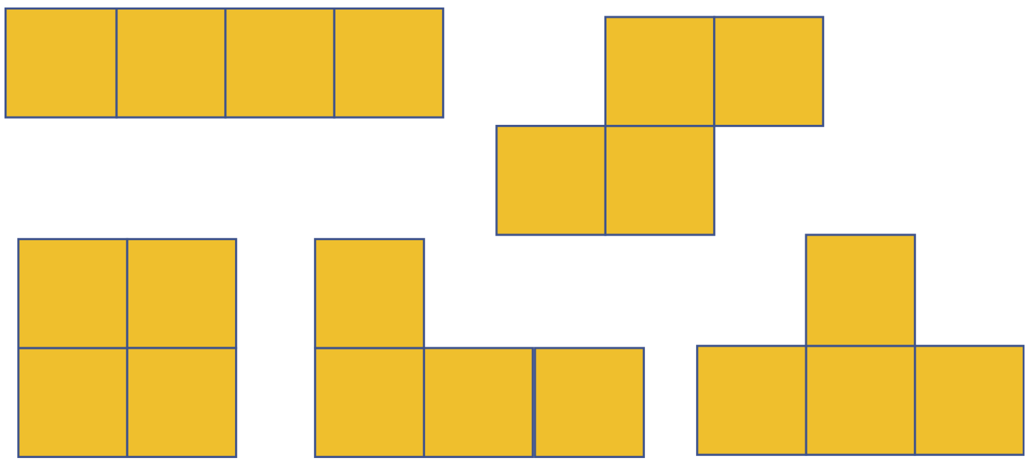


Figure 12: Polyominos constructed from 4 squares

The *order* of a polyomino is the minimum number of copies of the polyomino that can tile a rectangle (assuming that can be done).

There are polyominos of order 4:

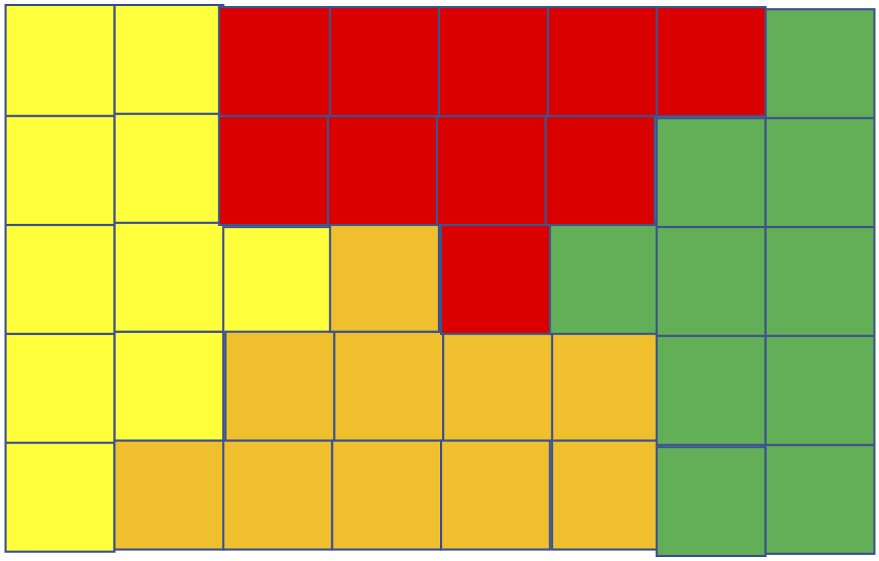


Figure 13: A rectangle tiled by 4 copies of a polyomino, no fewer copies of which tile a rectangle

There is no polyomino of order 3.

Is there a polyomino of order 5?

4.4 Runs of 0s in the binary expansion of the square root of 2

A problem of Paul Erdös asks if there are there arbitrarily long sequences of 0's in the binary expansion of $\sqrt{2}$.

4.5 Erdös–Straus conjecture

Is it true that for every positive integer n there are positive integers a, b, c such that

$$\frac{4}{n} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

See here for more details.

4.6 Catalan pseudo-primes

Are there any Catalan pseudo-primes other than 5907, 1194649, and 12327121?

See the original article that defined and discovered Catalan pseudo-primes:

Aebi, Christian, and Grant Cairns. Catalan numbers, primes, and twin primes. *Elemente der Mathematik* 63, no. 4 (2008): 153-164.

4.7 Cycles in cubic graphs

A connected graph is cubic if all its vertices have degree 3.

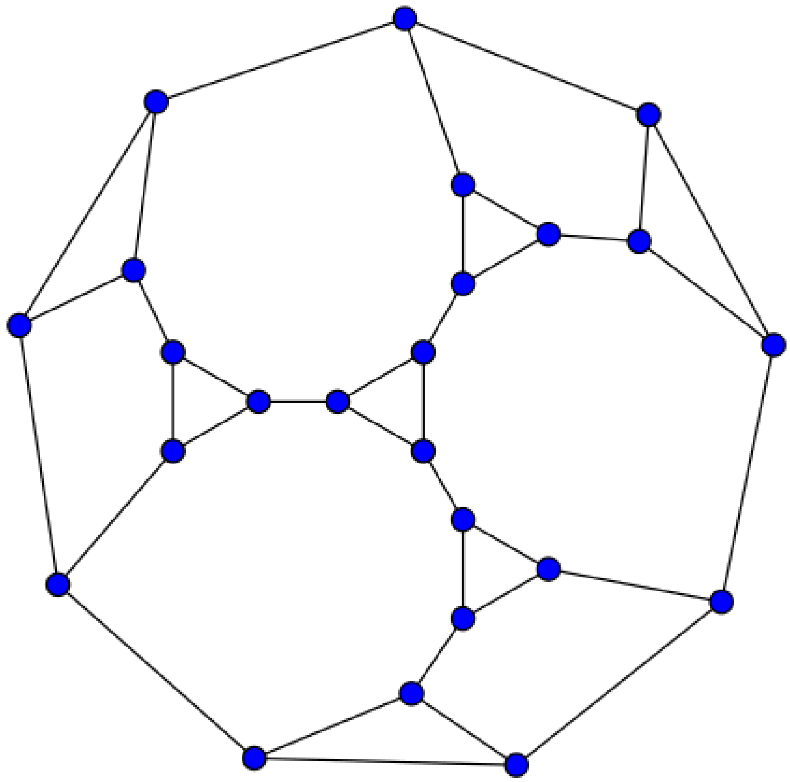


Figure 14: A cubic graph

A special case of the Erdős–Gyárfás conjecture is that every cubic graph contains a cycle of length a power of 2.

The cubic graph shown above has no cycles of length 2,4 or 8, but does have cycles of length 16:

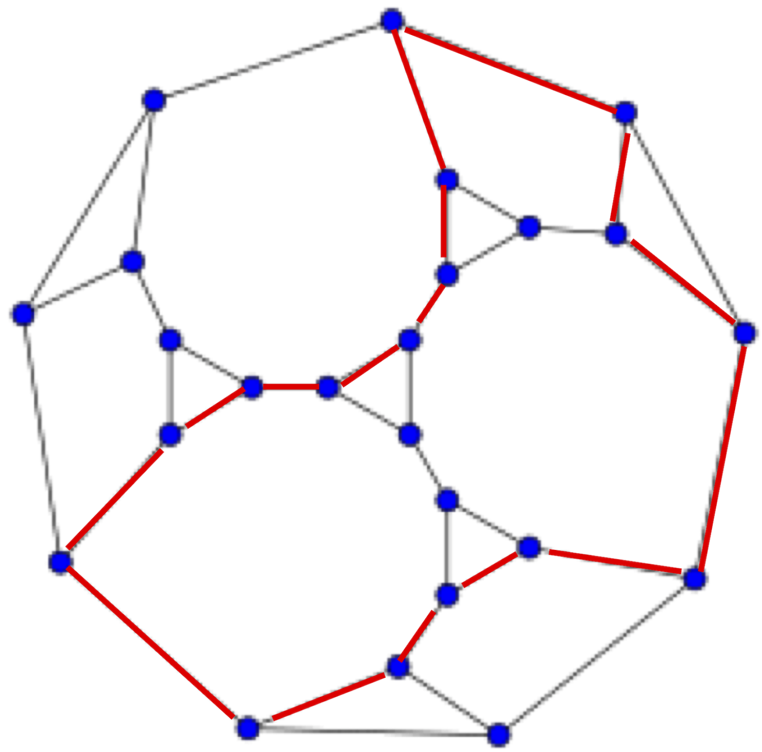


Figure 15: A cycle of length $2^4 = 16$, shown red

4.8 How many factorials modulo a prime?

For a non-negative integer n the factorial $n!$ is defined inductively as:

- $0! = 1$
- $n! = n \times (n-1)!$

It is an open problem to determine, for all prime numbers p, the size of the set

$$A(p) := \{k! (\text{mod } p) : k = 0, 1, \ldots, p-1\}$$

For example, for $p = 13$

$$A(p) = A(13) = \{1, 2, 3, 5, 6, 7, 9, 11, 12\}$$

which has size 9.

Investigate how the size of $A(p)$ varies with the prime p.

Also try to estimate the *average value* of $A(p)$ - that is, for a given prime p, calculate the size of $A(k)$ for primes $k \leq p$ and average those values. Estimate how that average varies with p.