Combinatorics, geometry, and number theory problems

Gary Davis & Adriano Marzullo

Contents

1 Introduction

In this document we have collected problems from combinatorics, combinatorial number theory, computational number theory, and geometry that are hopefully understandable, engaging and challenging for High School students and mathematics undergraduates and, with the exception of the unsolved problems, do-able by them.

Our main aim in collecting and presenting these problems is to provide students with a fun collection: problems that stimulate and challenge their mathematical thinking, and are fun to work on.

1.1 Several flavors of problems

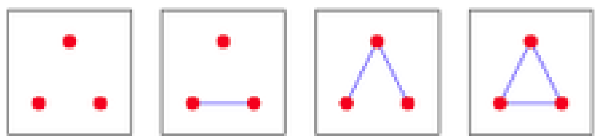
These problems tend to come in several distinct "flavors". One flavor consists of curious, often tantalizing, yet somewhat contrived, puzzles. These are often engaging, fun to work on, and sometimes offer beautiful insights. Working on these problems can be very satisfying and great fun.

Another flavor consists of *investigations*, in which there is not a single clearly stated problem, but instead a proposed investigation of some curious phenomenon.

Thirdly, another, generally very hard, flavor of problems consists of *unsolved problems*. These are mostly problems that several, if not many, professional mathematicians have thought about long and hard and have not yet been able to resolve. While it's sensible to be aware of these unsolved problems, and perhaps to think for a short time what the problem entails, it is generally not a wise move to spend a long time thinking about these hard unsolved problems simply because the chance of solving them is very low, and time spent on them means time taken away from other potentially solvable problems.

Fourthly, we have added a favor of approachable research problems. These are problems that are relatively easy to explain and understand, for which no-one at the time of writing knows an answer, but are approachable and potentially solvable by adaptable students.

Another flavor, which we will into go into in detail here consists of problems that come from other branches of mathematics. For example, in graph theory, a fundamental problem is how many essentially different graphs there are with a given number of vertices. For example, here are the 4 essentially different graphs with 3 vertices:



Counting, or "enumerating", the number of essentially different graphs with n vertices is a fundamental problem in graph theory and utilizes a method known as Polya-Burnside enumeration.

1.2 Software for computation

Occasionally you might need, or want to, use mathematical software to carry out computations. Two open source software packages for this purpose are:

1. SageMath
2. Geogebra

1.3 New delights and unexpected connections

As you peruse these problems and investigations bear in mind the words of the late, great Bill Thurston,

Figure 1: William P. Thurston

who wrote:

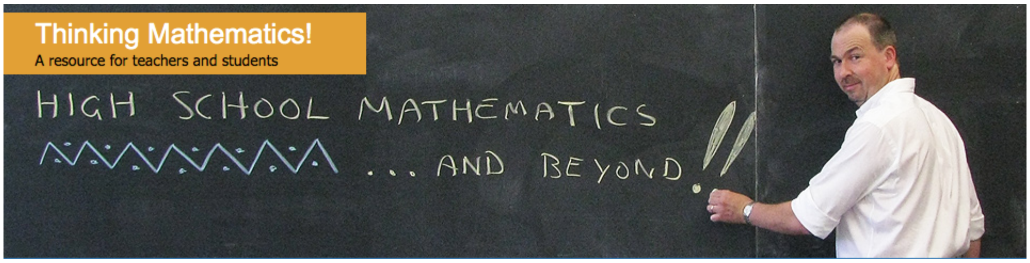
> "Mathematics has a remarkable beauty, power, and coherence, more than we could have ever expected. It is always changing, as we turn new corners and discover new delights and unexpected connections with old familiar grounds."

As you engage with a problem, and potentially get stuck and confused, bear in mind some other helpful words from Bill Thurston:

“Mathematics is a process of staring hard enough with enough perseverance at the fog of muddle and confusion to eventually break through to improved clarity. I’m happy when I can admit, at least to myself, that my thinking is muddled, and I try to overcome the embarrassment that I might reveal ignorance or confusion. Over the years, this has helped me develop clarity in some things, but I remain muddled in many others. I enjoy questions that seem honest, even when they admit or reveal confusion, in preference to questions that appear designed to project sophistication.”

1.4 Acknowledgements

A major stimulus and source of inspiration for us in compiling this list of mathematical problems for High School students and undergraduates, has been Dr. James Tanton (On Twitter: @jamestanton)



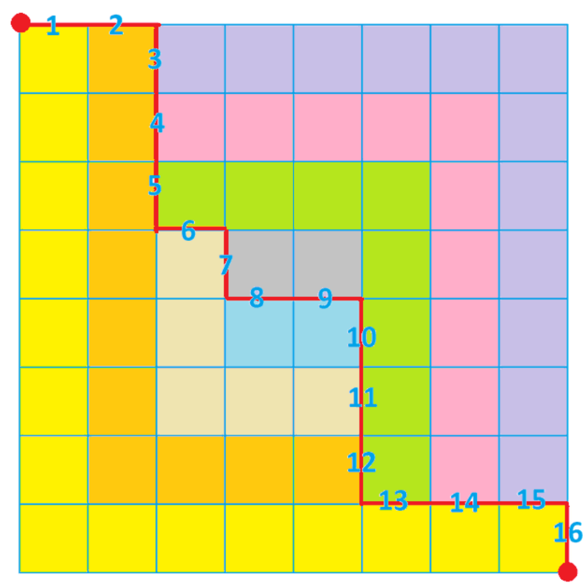
and you will see numerous acknowledgements to James on individual problems, where he contributed either the whole problem itself, or the stimulus for us to enlarge an original idea of his.

2 Curious, tantalizing puzzles

2.1 Coloring a square grid

This problem is from Klee, S., Malkin, K. & Pevtsova, J. (2021). Math Out Loud: An Oral Olympiad Handbook (Vol. 27). American Mathematical Society, and was posted on Twitter by James Tanton (@jamestanton):

Draw a path of horizontal and vertical steps from the top left corner to the bottom right corner of an $n \times n$ grid. If step 1 is horizontal, color all cells below it, turn 90 degrees, color rightward all squares back to the path. If step 1 is vertical, go rightward, then down. Do this for the first n steps. Is the whole grid sure to be colored?



2.2 Which integer sequences are possible?

(1) Sequences $a_0, a_1, a_2, a_3, \ldots$ of non-negative integers a_n are constructed subject to the following constraints:

- $a_0 = 0$
- $a_1 = 1$
- for all $n \geq 1, a_n = \lfloor \frac{a_{n-1}+a_{n+1}}{2} \rfloor$, where for a real number x, $\lfloor x \rfloor$ is the *floor* of x: the greatest integer less than, or equal to, x.

What are the possible integer sequences that satisfy these constraints?

Thanks to James Tanton (@jamestanton on Twitter) for this idea.

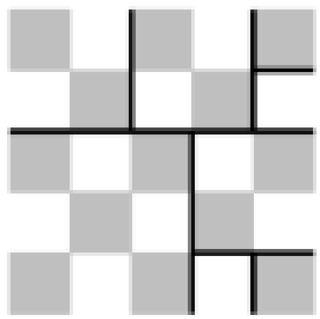
(2) As a variant, investigate and characterize positive integer sequences $a_1, a_2, a_3, \ldots$ constructed subject to the following constraints:

- $a_1 = 1$
- $a_2 = 1$
- for all $n \geq 1, a_n = \lfloor \sqrt{a_{n-1} * a_{n+1}} \rfloor$, where, as before, for a real number x, $\lfloor x \rfloor$ is the *floor* of x: the greatest integer less than, or equal to, x.

2.3 Cutting a checker board

This problem is adapted from Klee, S., Malkin, K. & Pevtsova, J. (2021). Math Out Loud: An Oral Olympiad Handbook (Vol. 27). American Mathematical Society:

A 5×5 checkerboard is cut along the grid-lines into a number of smaller square boards. Prove that the total length of the cuts is a multiple of 4.



What about a 25×25 checkerboard?

For which n is this true for an $n \times n$ checkerboard?

Is this true, for example, for n even?

2.4 Can Robert stop Kayley?

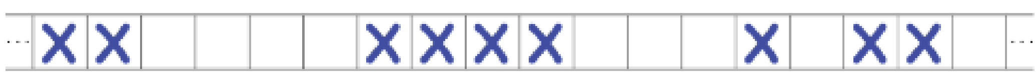
This problem is adapted from Klee, S., Malkin, K. & Pevtsova, J. (2021). Math Out Loud: An Oral Olympiad Handbook (Vol. 27). American Mathematical Society:

Two players, Kayley and Robert take turns in making and erasing "X" marks on an infinite tape of squares, as follows:

1. Kayley goes first, and at each turn marks 2 of the squares of the tape (not necessarily adjacent) with an "X"
2. Robert takes turns after Kayley and can erase any block of adjacent X's

What length blocks of adjacent X's can Kayley make without being stopped by Robert?

In the picture below there are blocks of length 1, 2 and 4 of adjacent X's:

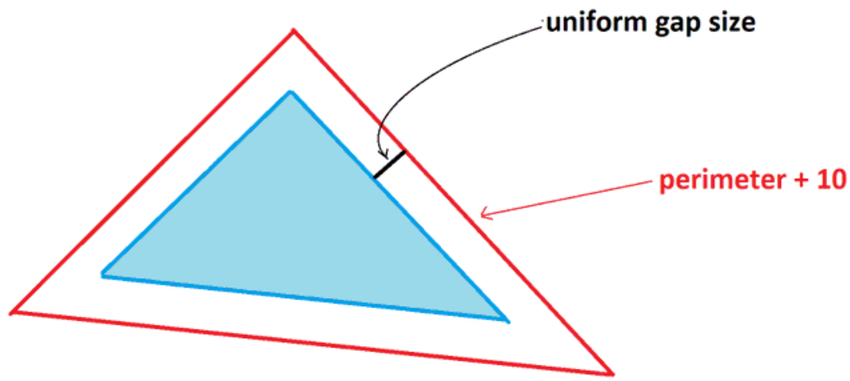


2.5 Wrapping a string around a polygon

A string, 10 units longer than the perimeter of a polygon, is wrapped around the polygon to make a similar polygon with a gap of uniform size between the two perimeters.

In terms of the area and the perimeter of the original polygon, how big is the gap?

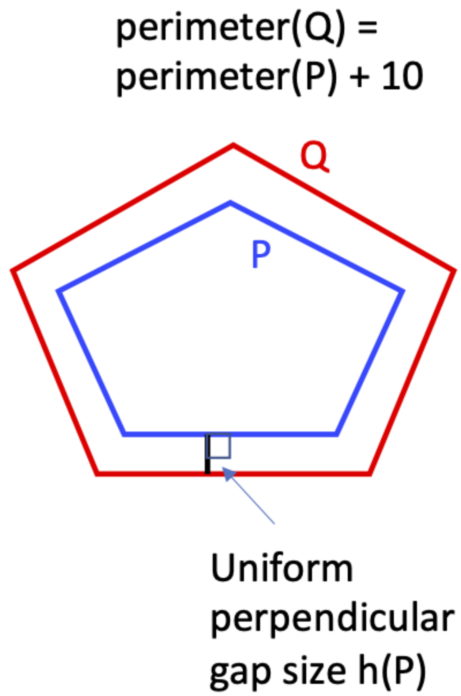
Here is a picture for the case of a triangle:



Does it make difference if the polygon is, or is not, convex?

For each convex polygon P define a number $h(P)$ as follows:

> as above, wrap a string of length perimeter$(P) + 10$ around P, maintaining a uniform perpendicular gap size, $h(P)$.



Call two convex polygons P_1, P_2 *gap-equivalent* if $h(P_1) = h(P_2)$

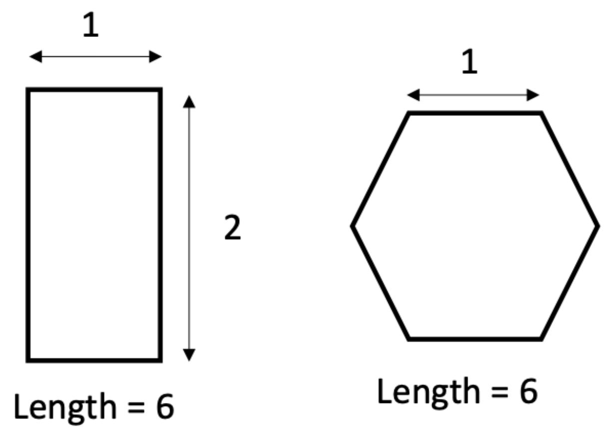
- Are all rectangles gap-equivalent?
- Are all right triangles gap-equivalent?
- Are similar polygons gap-equivalent?

Thanks for these problems goes to James Tanton who raises an interesting question:

For a convex polygon P with perimeter p and area A, is P tangential - that is, P has an inscribed circle - if and only if $h(P) = 20A/p^2$?

2.6 Covering a polygon by a disk

Define the "length" of a polygon to be the sum of the lengths of all its edges.



In terms of the length, L, of a polygon P, what is the radius of a smallest disk that entirely covers P?

Thanks to Igor Pak for this problem.

2.7 Tiling with tetrominos

This problem is from James Propp.

Figure 2: James Propp

An Aztec diamond of order n is

> "a region composed of $2n(n+1)$ unit squares, arranged as a stack of $2n$ centered rows of squares, with the k^{th} row having length $\min(2k, 4n-2k+2)$"

Jockusch, W., Propp, J., & Shor, P. (1998). Random domino tilings and the arctic circle theorem

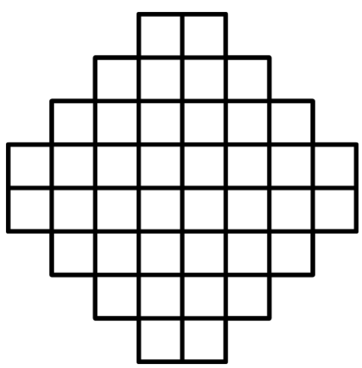


Figure 3: Aztec diamond of order 4

We are going to try to tile Aztec diamonds with tetrominos.

The 5 different tetrominos are:

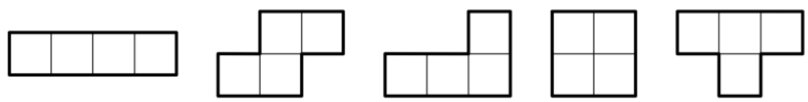


Figure 4: The 5 distinct tetrominos

Here is a tiling of the Aztec diamond of order 3 by tetrominos:

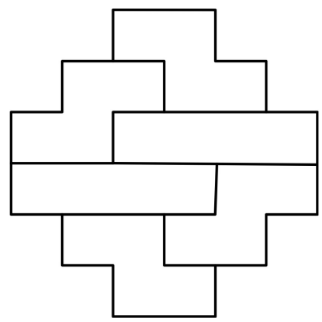
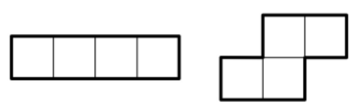


Figure 5: Aztec diamond of order 3 tiled by tetrominos

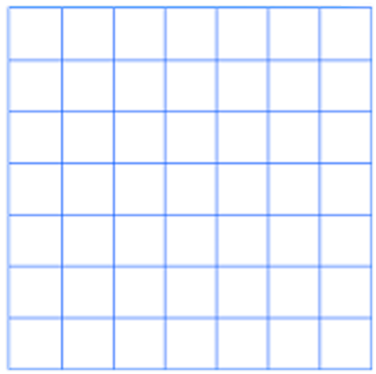
For which values of n can an Aztec diamond of order n be tiled by the first two tetrominos to the left of Figure 4 (the tetrominos shown below)?



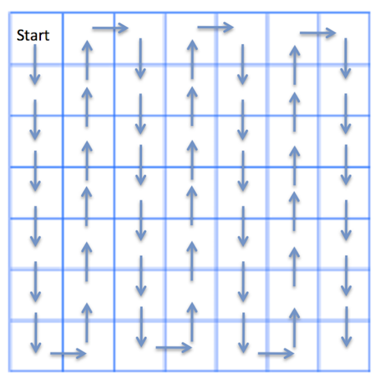
Another tiling problem: is it possible to tile a *rectangle* using all five tetrominos?

2.8 Grid walks

On the 7×7 grid shown below:

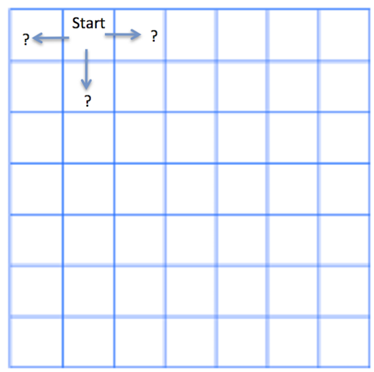


a *walk* from a START square of the grid, consists of a sequence of squares of the grid, beginning with START, and where each succeeding square is adjacent, horizontally or vertically, to the preceding square in the sequence:



The walk shown above passes through each square of the grid once and only once.

From the START square shown below, is it possible to walk so as to reach each square of the grid once and only once. If so, show how, and if not, explain why not:



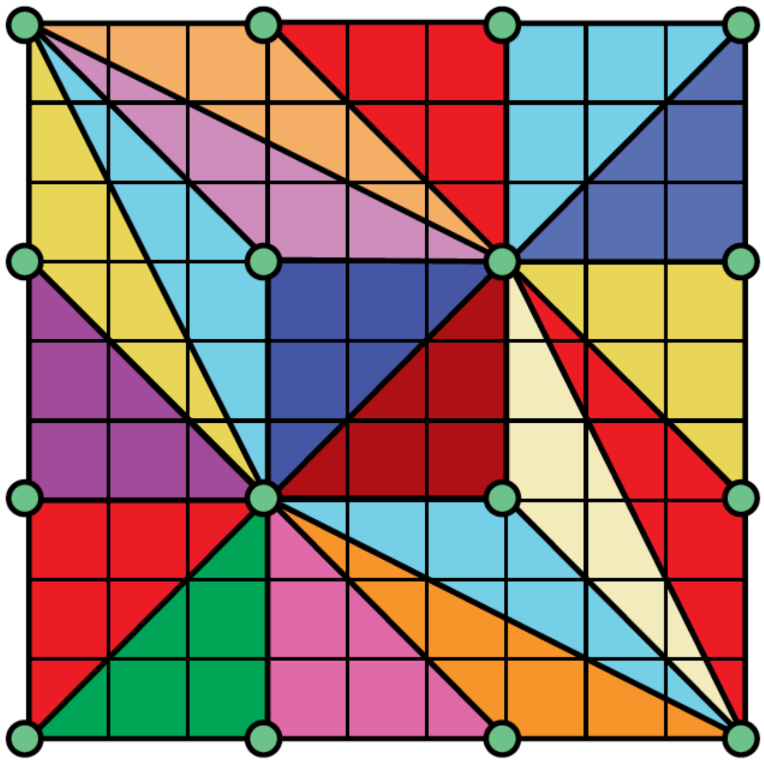
From which squares of the 7×7 grid is it possible to start a walk that reaches each square of the grid once and only once?

What if the 7×7 grid is replaced by an $n \times n$ grid for some other integer n?

Thanks to James Tanton for opening our eyes to this problem - watch James introduce it with Sunil Singh on YouTube.

2.9 Subdividing a square into triangles

It is possible to divide a nine-by-nine grid of squares into 18 triangles of equal area, each with a vertex at an intersection point on the grid:



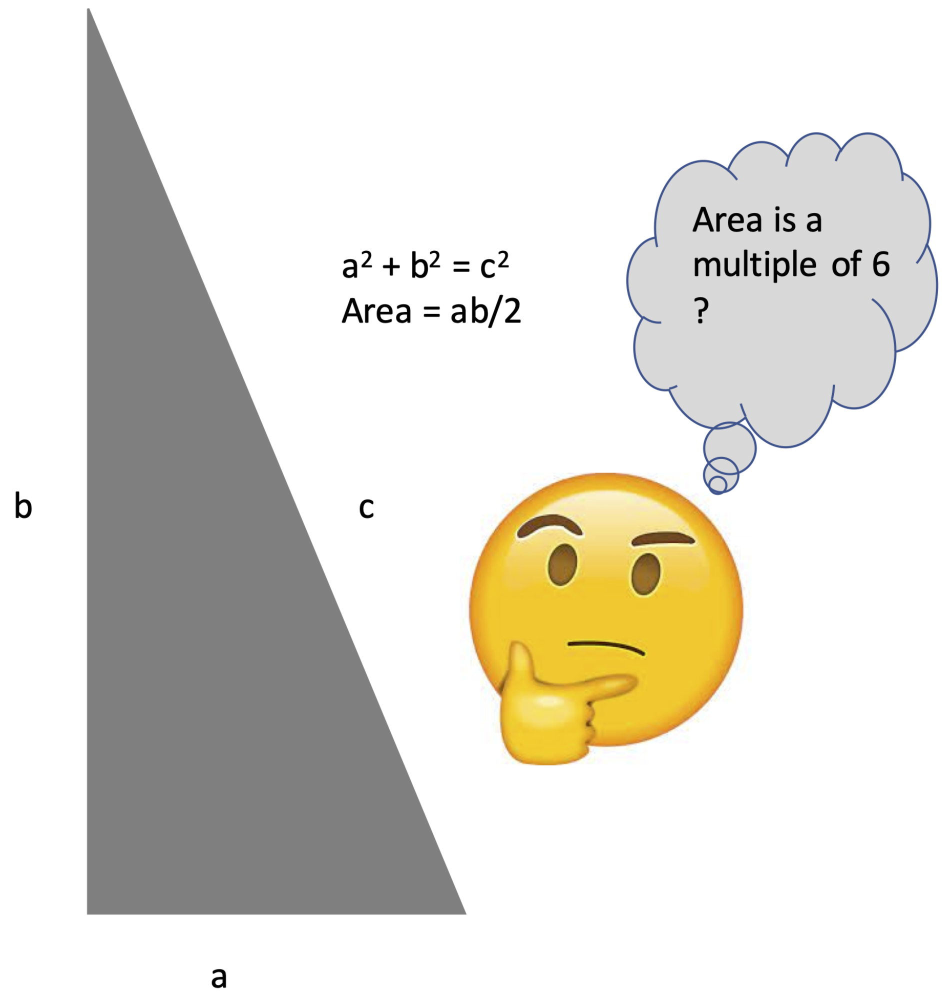
Is it possible to divide the nine-by-nine grid of squares into an odd number of triangles of equal area (with vertices at grid points)?

This problem is from James Tanton (@jamestanton on Twitter).

What about $n \times n$ grids? Does it make a difference if n is odd or even?

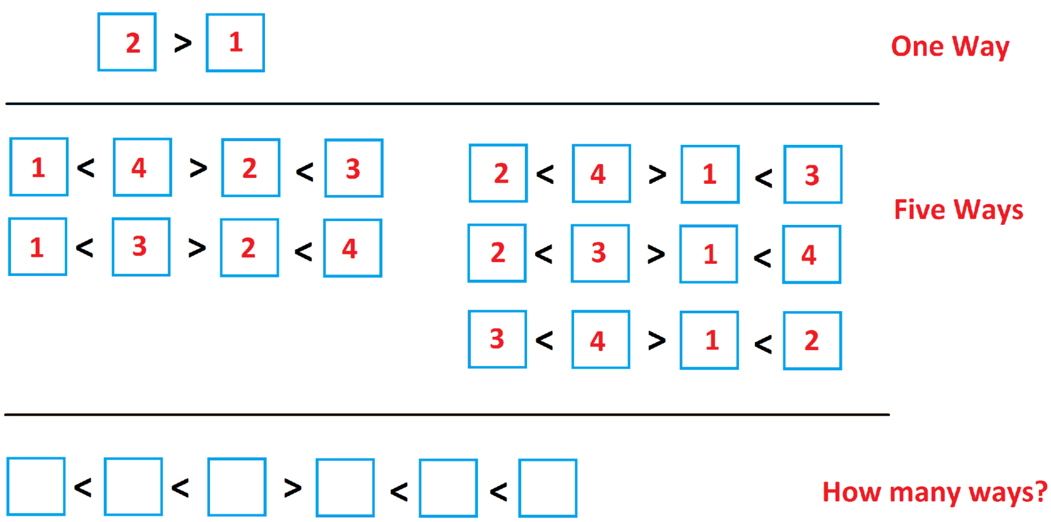
2.10 Area of integer right triangles

Can you prove conclusively that the area of any right triangle with integer side-lengths is sure to be a multiple of 6?



2.11 Arranging the numbers 1, 2, ... , 2n-1, 2n

In how many ways can one arrange the numbers 1, 2, 3, 4, 5, 6 and respect the inequalities between adjacent terms as shown? (All inequalities are < except for the middle one.)



How about 1, 2, 3, 4, 5, 6, 7, 8? Or 1, 2, 3, 4, 5, 6, 7, 8, 9, 10?

Thanks to James Tanton for this problem.

2.12 How many sixes from throwing many dice?

Imagine we throw n dice all at once, and record how many 6's we see.

As a function of n,

- What is the probability of getting an even number of 6's?
- What is the probability of getting an odd number of 6's?
- What is the probability of getting a number of 6's that leaves a remainder of 0 (mod 3)?
- What is the probability of getting a number of 6's that leaves a remainder of 1 (mod 3)?
- What is the probability of getting a number of 6's that leaves a remainder of 2 (mod 3)?

Thanks to James Tanton for this problem.

2.13 Pythagorean triples

If (a, b, c) is a Pythagorean triple - meaning a, b, c are positive integers and $a^2 + b^2 = c^2$ - can you prove conclusively that:

- at least one of a or b is sure to be divisible by 3
- at least one of a or b is sure to be divisible by 4, and
- at least one of a, b, or c is sure to be divisible by 5

?

$$a^2 + b^2 = c^2$$

(3,4,5) (6,8,10) (7,24,25)

(5,12,13) (20,21,29) (8,15,17)

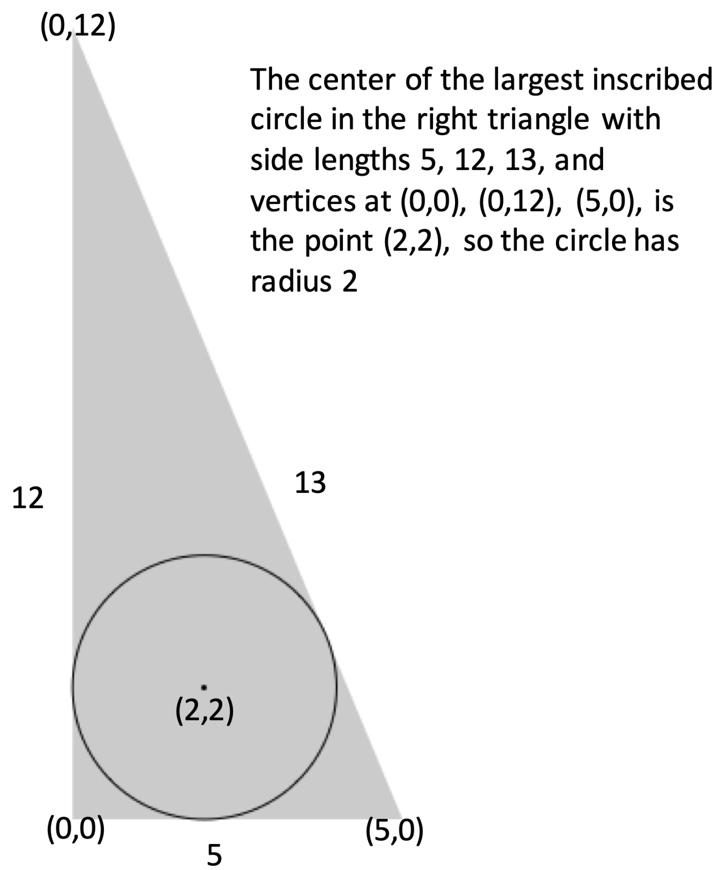
(20,99,101) (48,55,73) (17,144,145)

Pythagorean Triples

Thanks to James Tanton (@jamestanton on Twitter) for this problem.

2.14 Inscribed circles in integer right triangles

In the right triangle below, with side lengths 5, 12, 13, the largest inscribed circle has integer radius:

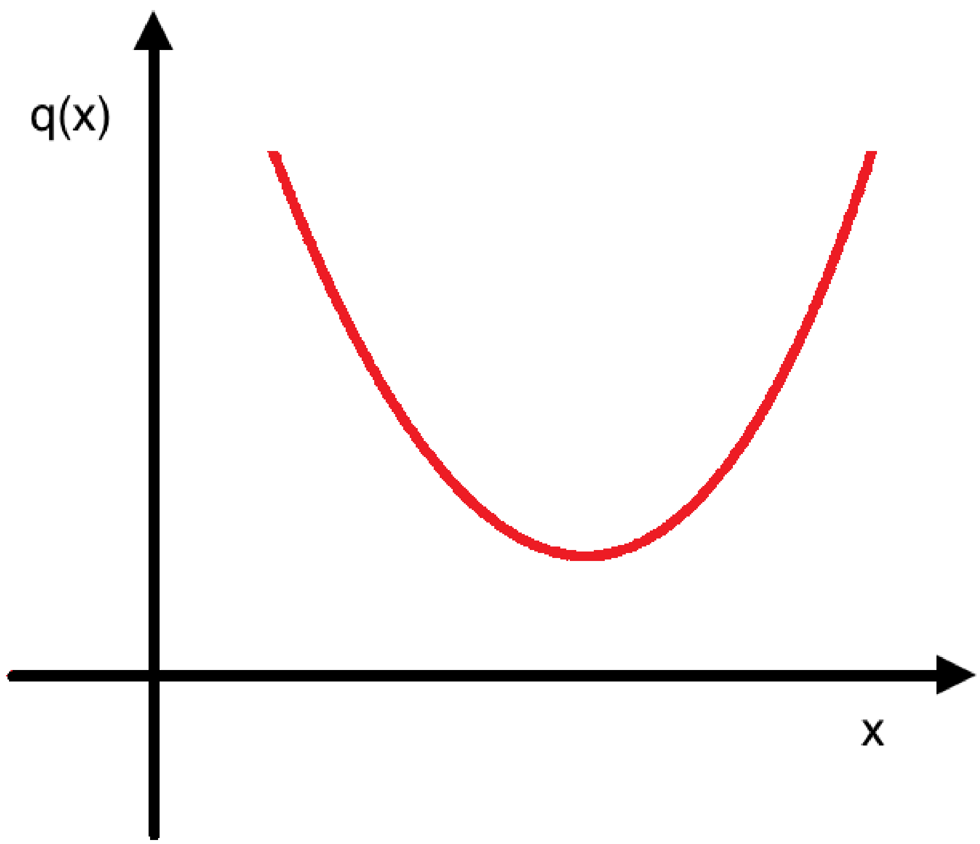


Can you prove conclusively that the radius of the largest circle one can draw inside any right triangle with integer side-lengths is sure to have an integer radius?

Thanks to James Tanton (@jamestanton on Twitter) for this problem.

2.15 Integer values of a quadratic

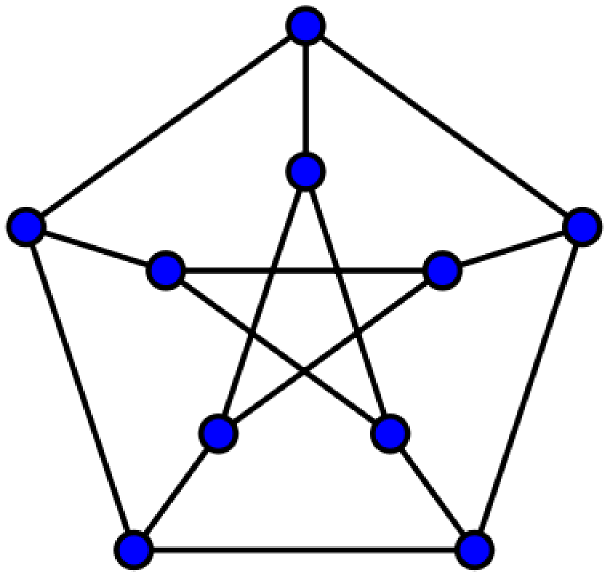
A quadratic $q(x) = ax^2 + bx + c$ has integer outputs for 3 distinct integer inputs. Must $q(x)$, in fact, be an integer for 4 integer inputs x? 5 integer inputs? Infinitely many integer inputs?



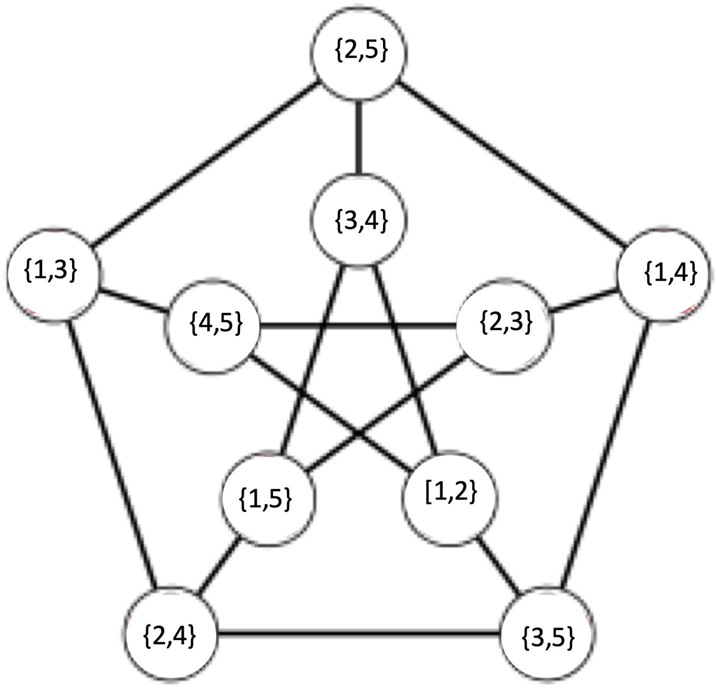
Thanks to James Tanton for this problem.

2.16 Symmetries of the Petersen graph

The Petersen graph is the following graph with 10 vertices and 15 edges:



We can label the vertices of the Petersen graph with pairs of integers chosen from $\{1, 2, 3, 4, 5\}$ so that two vertices are joined by an edge if and only if their labels are disjoint (that is, have no numbers in common):



A permutation of the set $\{1, 2, 3, 4, 5\}$ is a rearrangement of that set. There are $5! = 120$ permutations of $\{1, 2, 3, 4, 5\}$.

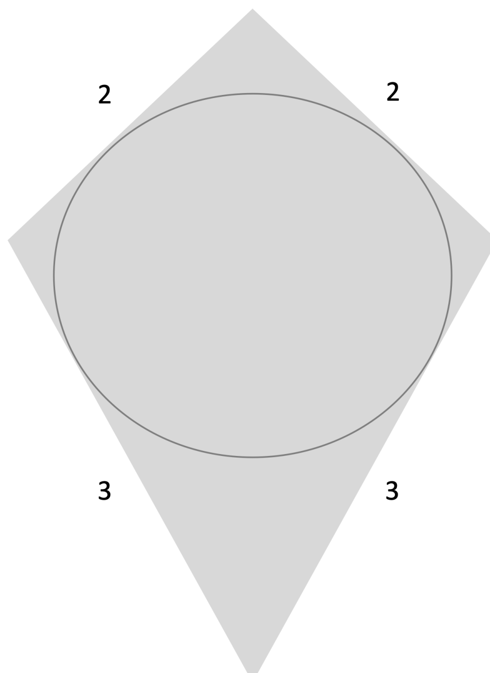
Show that every permutation of $\{1, 2, 3, 4, 5\}$ gives a symmetry of the Petersen graph: each permutation maps the labels of a vertex to a new label so that two vertices are joined by an edge if and only if the permuted labels are joined by an edge.

Can you show there are no other symmetries of the Petersen graph?

2.17 Quadrilaterals with an inscribed circle

A convex quadrilateral Q is constructed such that:

- Each of the 4 side lengths of Q is one of the integers $1, 2, 3$. Note: repetitions are allowed so some, or all, of the sides will be of equal length.
- Q contains a circle tangent to each of the 4 sides.



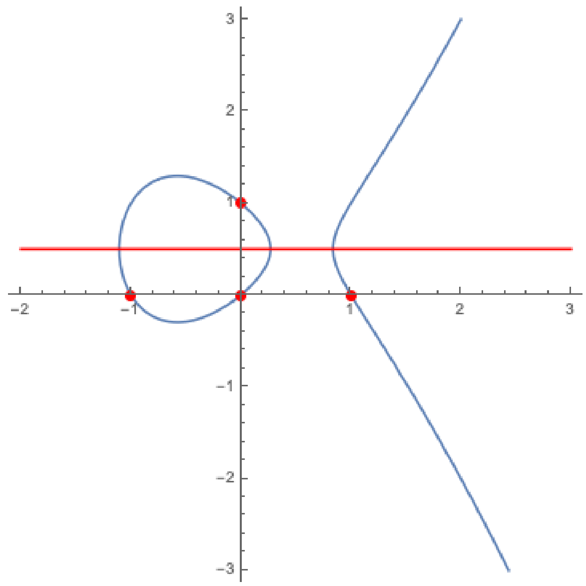
How many such quadrilaterals are there, and what are their 4 side lengths L_1, L_2, L_3, L_4?

What are the centers and radii of their inscribed circles?

What are the areas of the quadrilaterals?

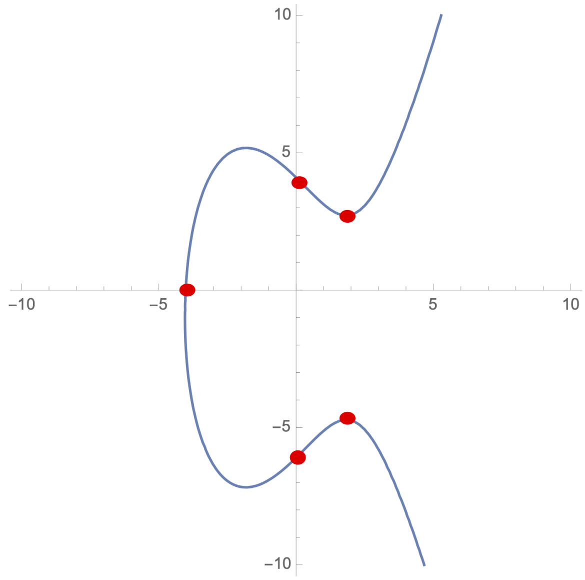
2.18 Features of elliptic curves

The set of points $E := \{(x, y) \in \mathbb{R}^2 : y^2 - y = x^3 - x\}$ is an example of an elliptic curve:



Find the red points and the red line - about which the curve is symmetric (why is the curve symmetric about that line?).

The elliptic curve $y^2 + 2y = x^3 - 10x + 25$ is shown below - find the indicated points on the curve:



A basic issue in the theory of elliptic curves is to find all points (x, y) on an elliptic curve such that both x, y are integers - so-called “integer points”. Can you find all the integer points on the above two elliptic curves?

The Elliptic Curve Plotter is a graphical application that illustrates elliptic curves. Users can sketch elliptic curves and experiment with their group law, and save images in PNG or SVG format for later use. Thanks to Adam Hausknecht for information about the elliptic curve plotter.

2.19 How many increasing trees?

An "*increasing tree*" is a tree with n vertices, labelled 1, 2, ..., n, with the root of the tree labelled "1", such that the vertex labels are *increasing* as we travel down the tree from the root:

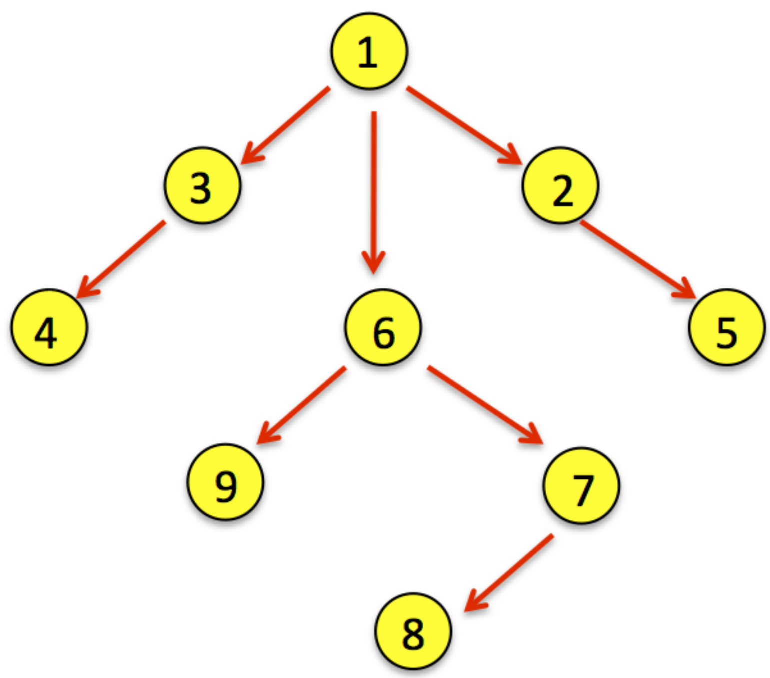


Figure 6: An increasing tree on 9 vertices

For each natural number n, how many increasing trees with n vertices are there?

Thanks to Per Alexandersson for this problem.

2.20 The Calkin-Wilf tree

The Calkin-Wilf tree is a binary tree with root $\frac{1}{1}$ and each entry $\frac{a}{b}$ branches into a l*eft child* $\frac{a}{a+b}$ and *right child* $\frac{a+b}{b}$:

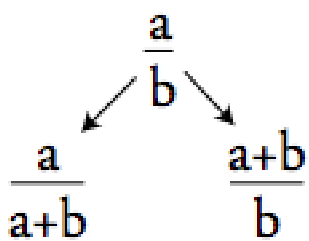


Figure 7: $\frac{a}{b}$ has left child $\frac{a}{a+b}$ and right child $\frac{a+b}{b}$

The rational numbers $\frac{a}{b}$ in the Calkin-Wilf tree occur in *levels*, where level 0 is $\{\frac{1}{1}\}$ and level n, for $n \geq 1$, consits of the left and right children of rational numbers in level $n-1$:

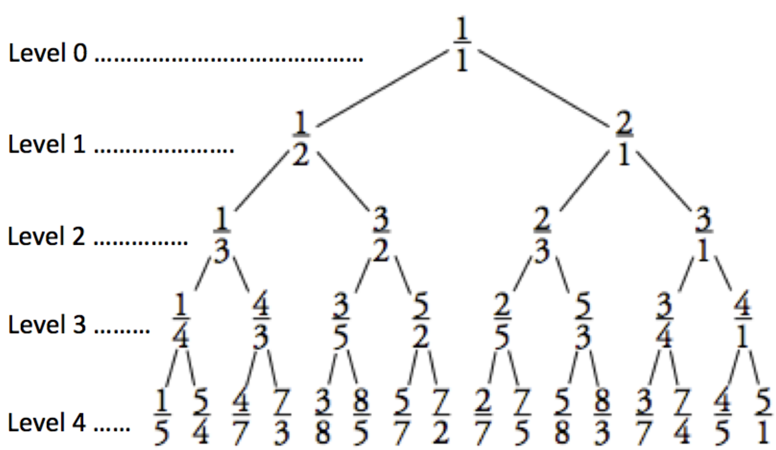


Figure 8: Levels 0 through 4 of the Calkin-Wilf tree

Can you prove the following about the Calkin-Wilf tree?

- For each entry $\frac{a}{b}$ in the Calkin-Wilf tree, $a \geq 1, b \geq 1$ and the greatest common divisor (GCD) of a and b is 1.
- Every rational number $\frac{a}{b}$ with $a \geq 1, b \geq 1$ and $\text{GCD}(a, b) = 1$ occurs once and only once in the Calkin-Wilf tree.
- The list, left to right, of denominators in level n of the Calkin-Wilf tree is the reverse of the numerators, left to right, in level n.
- There are 2^n terms in level n.
- The sum of the numerators (= the sum of the denominators) in level n is 3^n.

Investigate a formula for the *average* value of all terms in level n of the Calkin-Wilf tree.

2.21 Average of triangular numbers

The n^{th} triangular number is $T(n) = \frac{n\times(n+1)}{2}$.

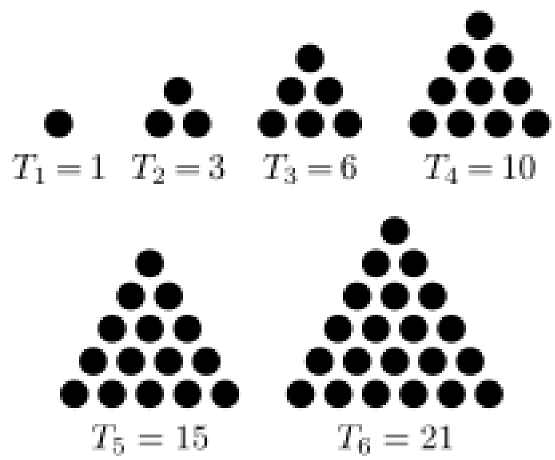


Figure 9: The first six triangular numbers

For which n is the average, $\frac{1}{n}\sum_{k=1}^{n} T(k)$, of the first n triangular numbers $T(1), T(2), \ldots, T(n)$ an integer?

2.22 Average of Stirling numbers

The unsigned Stirling number of the first kind, denoted $S(n,k)$ where $0 \leq k \leq n$, can be defined recursively as follows:

- $S(0,0) = 1$
- $S(0,n) = 0 = S(n,0)$
- $S(n,k) = (n-1)S(n-1,k) + S(n-1,k-1)$ for $n \geq 1$

Below is a table of $S(n,k)$ for $0 \leq n \leq 10$:

n \ k	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	0	1									
2	0	1	1								
3	0	2	3	1							
4	0	6	11	6	1						
5	0	24	50	35	10	1					
6	0	120	274	225	85	15	1				
7	0	720	1764	1624	735	175	21	1			
8	0	5040	13068	13132	6769	1960	322	28	1		
9	0	40320	109584	118124	67284	22449	4536	546	36	1	
10	0	362880	1026576	1172700	723680	269325	63273	9450	870	45	1

For $n \geq 1$ let $A(n)$ denote the average, $\frac{1}{n}\sum_{k=1}^{n} S(n,k)$, of the Stirling numbers $S(n,1), S(n,2), \ldots, S(n,n)$.

Is $A(n)$ always an integer?

Formulate, and try to prove, a formula for $A(n)$ as a function of n.

2.23 Infinitely many 3's in this sequence?

In a Twitter post, James Tanton (@jamestanton) defined the sequence

$$s(0), s(1), s(2), s(3), s(4), \ldots$$

as follows:

- $s(0) = 0$
- $s(2n + 1) = 0$
- $s(2n) = 3s(n) + s(n - 1)$

for $n \geq 1$, and asked: is $s(n) = 3$ for infinitely many n?

The first 100 values of $s(n)$ are:

0, 1, 3, 0, 10, 0, 3, 0, 30, 0, 10, 0, 9, 0, 3, 0, 90, 0, 30, 0, 30, 0, 10, 0, 27, 0, 9, 0, 9, 0, 3, 0, 270, 0, 90, 0, 90, 0, 30, 0, 90, 0, 30, 0, 30, 0, 10, 0, 81, 0, 27, 0, 27, 0, 9, 0, 27, 0, 9, 0, 9, 0, 3, 0, 810, 0, 270, 0, 270, 0, 90, 0, 270, 0, 90, 0, 90, 0, 30, 0, 270, 0, 90, 0, 90, 0, 30, 0, 90, 0, 30, 0, 30, 0, 10, 0, 243, 0, 81, 0, 8

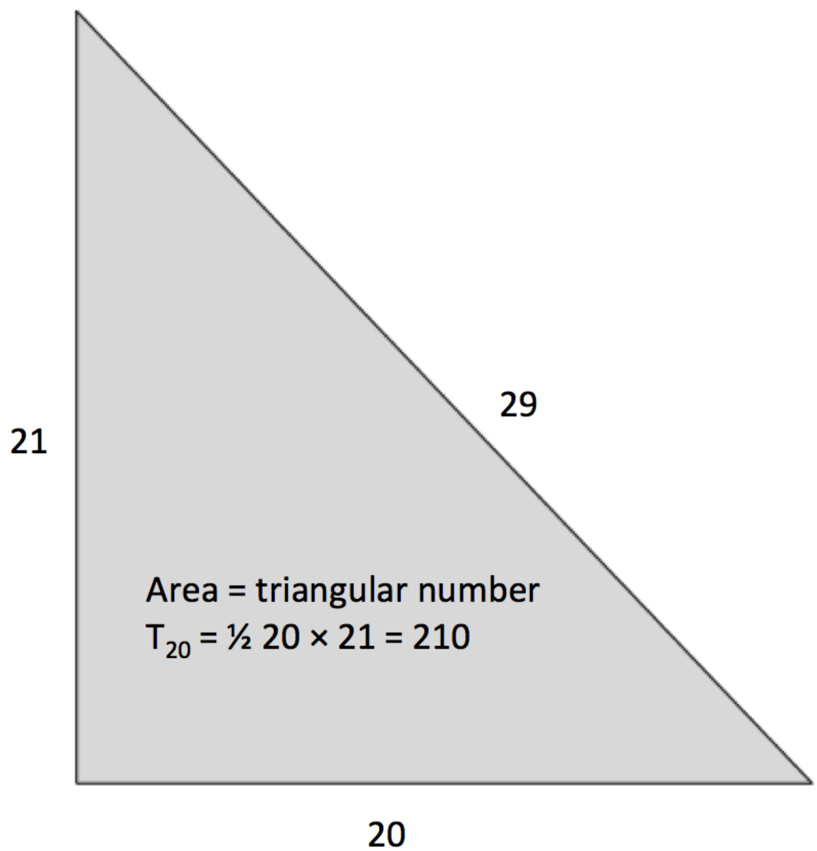
2.24 Sum of squares of consecutive integers

We know that $3^2 + 4^2 = 5^2$.

What is probably not so well known is that $20^2 + 21^2 = 29^2$.

For which other positive integers n is it true that $n^2 + (n + 1)^2$ is the square of a positive integer?

Is there some way to recursively determine such positive integers n?



2.25 A prime coincidence?

Ciara had learned from her mathematics teacher about modular arithmetic, and was especially intrigued by arithmetic in the set $\mathbb{Z}_p = \{0, 1, 2, \ldots, p-1\}$ for p a prime number.

Ciara was fascinated that it was possible to do division by non-zero elements of $\mathbb{Z}_p$ because, thanks to the fact that p is prime, for every $0 \neq x \in \mathbb{Z}_p$ there is an "inverse" $0 \neq y \in \mathbb{Z}_p$ for which $xy \equiv 1(\text{mod p})$.

Ciara wrote a computer program that, given a prime number p, would print out the inverse of each non-zero element of $\mathbb{Z}_p$.

x	inverse of x, mod 13
1	1
2	7
3	9
4	10
5	8
6	11
7	2
8	5
9	3
10	4
11	6
12	12

Being in a playful mood, Ciara calculated the difference between x and inverse of x, mod p for each $x \in \mathbb{Z}_p$ and then formed the sum of all these numbers:

x	inverse of x, mod 13	x-inverse of x, mod 13
1	1	0
2	7	8
3	9	7
4	10	7
5	8	10
6	11	8
7	2	5
8	5	3
9	3	6
10	4	6
11	6	5
12	12	0

The total of all $x - inv(x)$ mod 13 is 65.

Ciara did these calculations for the first 25 odd primes p (primes other than 2) and found that the total was always $\frac{p(p-3)}{2}$:

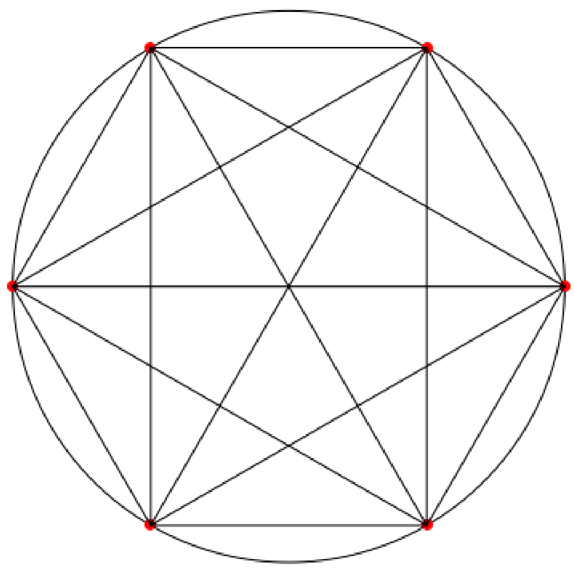
odd prime p	Ciara's total for p	p(p-3)/2
3	0	0
5	5	5
7	14	14
11	44	44
13	65	65
17	119	119
19	152	152
23	230	230
29	377	377
31	434	434
37	629	629
41	779	779
43	860	860
47	1034	1034
53	1325	1325
59	1652	1652
61	1769	1769
67	2144	2144
71	2414	2414
73	2555	2555
79	3002	3002
83	3320	3320
89	3827	3827
97	4559	4559

Could this be just a coincidence?

If not, why might it be true?

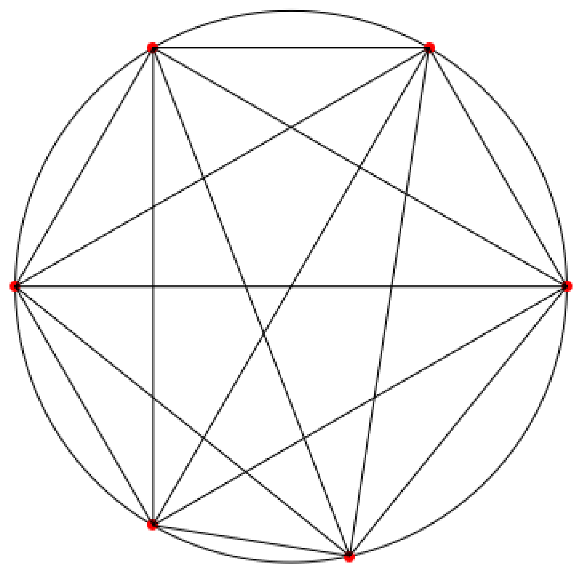
2.26 Lines dividing a circular disk into regions

In the picture shown below, 6 points are arranged regularly around a circle - going anti-clockwise around the circle, the distance between one point and the next is always the same:



When each pair of points is joined by a line, the lines divide the circular disk into 30 regions (count them!).

Yet, we can place 6 points a little irregularly around the circle and when lines are drawn between all distinct pairs of points we might get the lines dividing the circular disk into 31 regions (count them!):



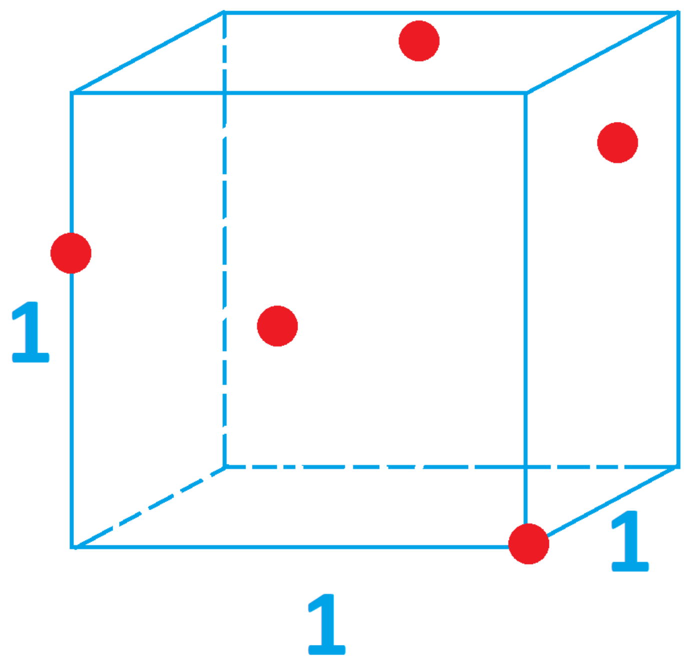
Can you prove that 31 is the *maximum* number of regions we can get by joining lines between 6 points on the circle?

Investigate the difference between the number of regions for regularly spaced points, and the maximum number of regions, for $n = 1, 2, 3, 4, 5, 6, 7, 8, 9, \ldots$ points.

Can you prove that the the number of regions for regularly spaced points, and the maximum number of regions, is the same for $n = 2, 4$ or any odd number?

2.27 Five points on a cube

Five points p_1, p_2, p_3, p_4, p_5 are placed on the surface of a cube with side length 1:



$D(p_i, p_j)$ is the distance, as measured along the surface of the cube between points $p_i \neq p_j$.

Denote by δ the minimum of the distances $D(p_i, p_j)$ for $p_i \neq p_j$:

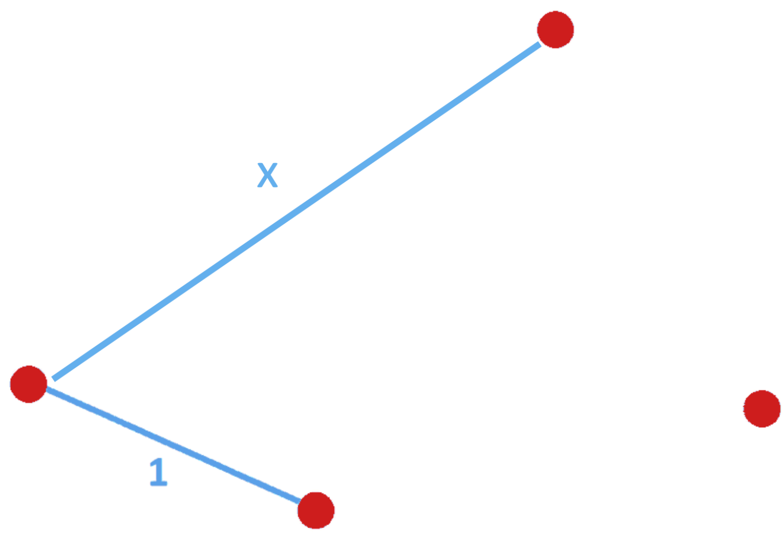
$$\delta = \min\{D(p_i, p_j) : 1 \leq p_i \neq p_j \leq 5\}$$

What is the maximum value of δ as the points p_1, p_2, p_3, p_4, p_5 vary over the surface of the cube?

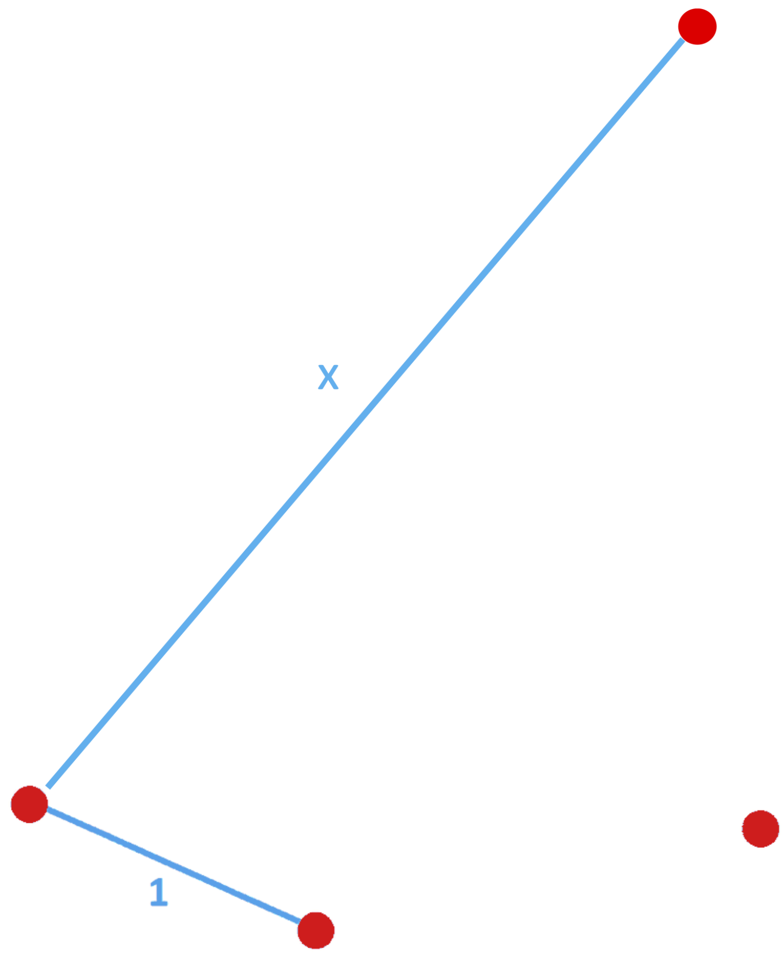
Thanks to James Tanton for this problem

2.28 Four points in the plane

There are four points in the plane. The *smallest* distance between a pair of points is 1 unit. The *largest* distance between a pair of points is x units.



Of course, x can be quite large as the next picture indicates:



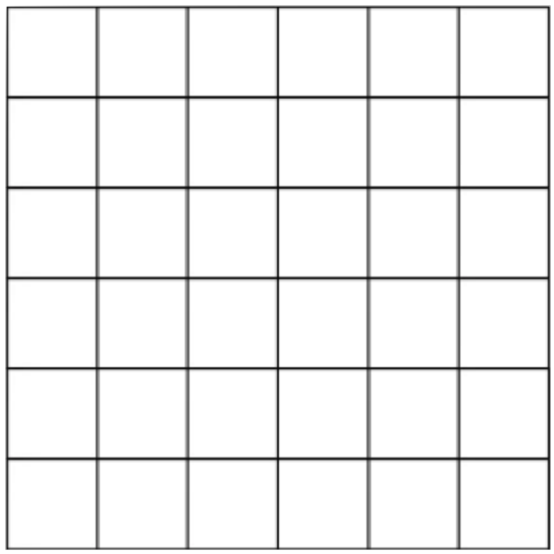
What's the *smallest possible* value of x?

Thanks to James Tanton for this problem

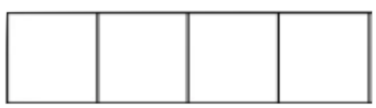
What if the 4 points were on a sphere of radius n and we measure distance on the surface of the sphere?

2.29 Tiling a square by rectangles

Can a 6×6 square grid, shown below:



be tiled by (nine) 4×1 rectangles:

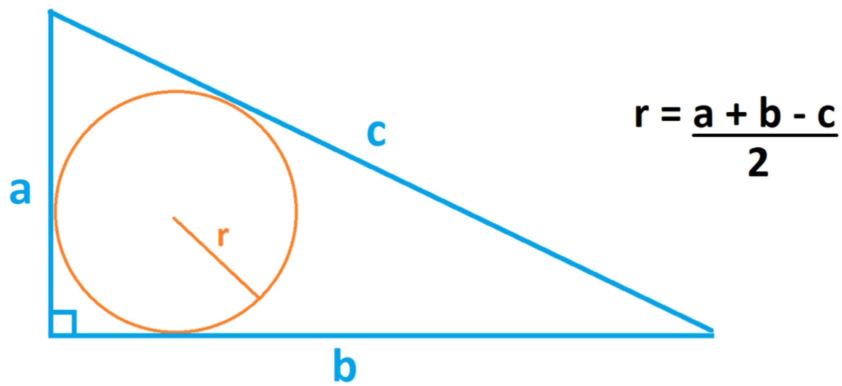


If so, how?

And if not, why not?

2.30 Incircle of a right triangle

A right triangle with integer side lengths a, b, c as shown below has a+b-c an even number. (Why is that?)



The radius r of its incircle is this even number divided by two. (Why is that?)

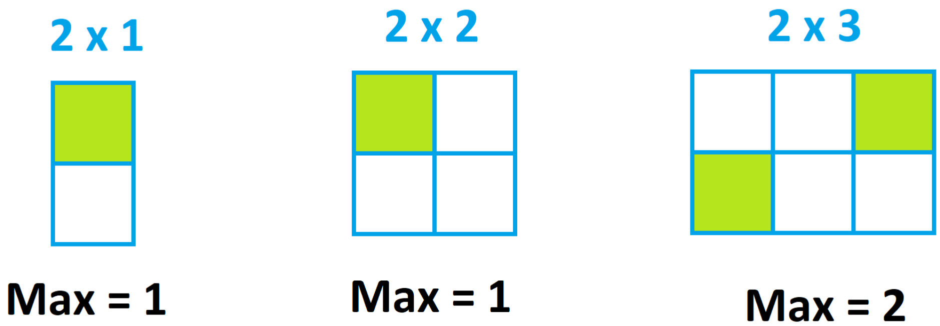
The 3-4-5 triangle has r=1; the 5-12-13 has r=2.

For each integer n, is there a primitive right triangle with r=n?

Thanks to James Tanton for this problem.

2.31 Coloring squares in a long, thin rectangular grid

Imagine a $2 \times n$ rectangular grid of squares, in which the maximum possible number of squares are colored so that no two colored squares touch, not even at the corners:



How does the maximum number of colored squares depend on n?

Thanks to James Tanton for this problem.

How would your answer differ for $3 \times n$ rectangular grids?

2.32 Coloring squares in a square grid

As a variant on the preceding problem of coloring squares in a long thin rectangle, James Tanton asks for the maximum number of squares that can be colored in an $n \times n$ grid of squares so that no two colored squares touch, not even at a corner:

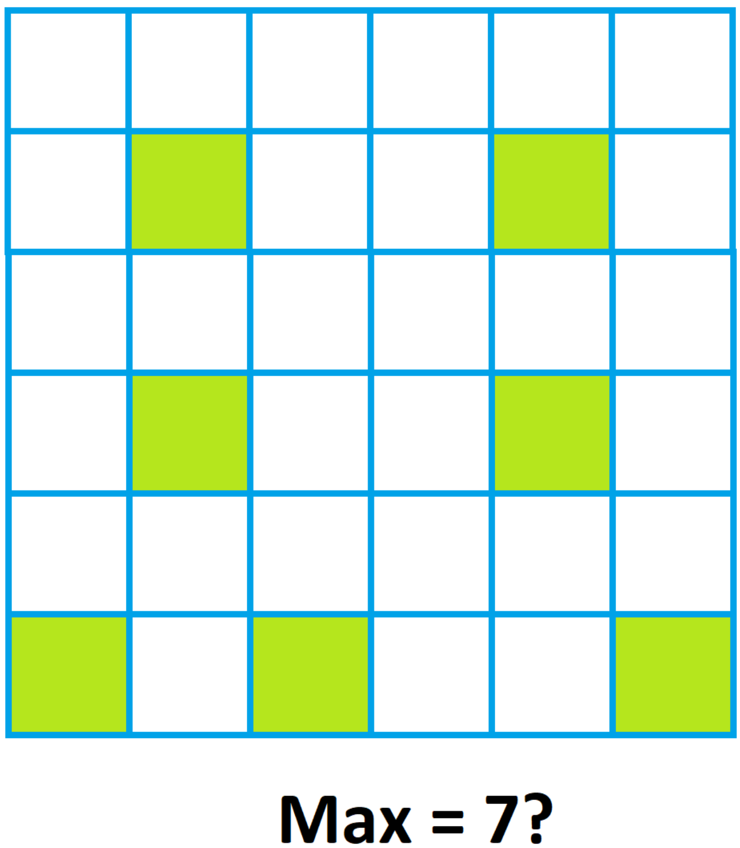


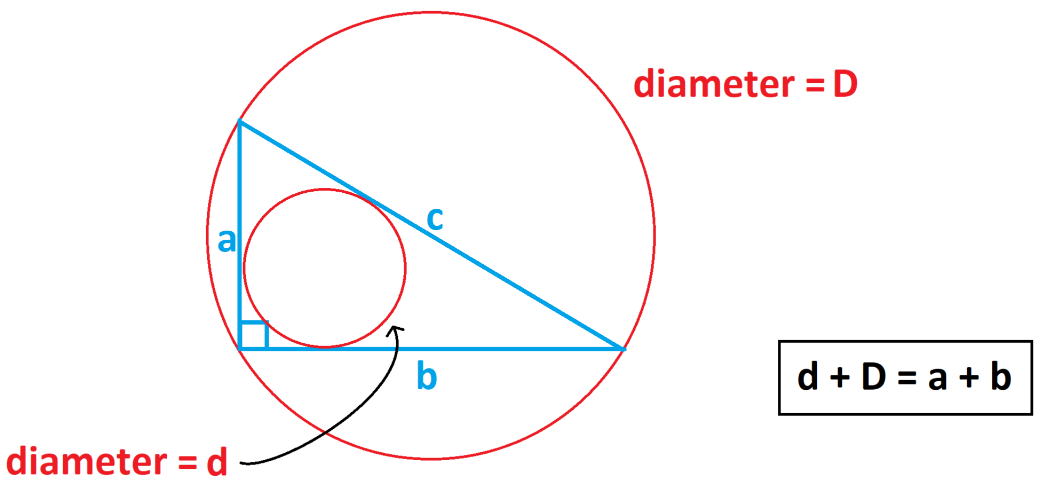
Figure 10: Is 7 the maximum number of colored squares that can be placed in a 6×6 grid?

2.33 Circumscribed and inscribed circles of a triangle

The circumscribed circle of a triangle is a circle that passes through all the vertices of the triangle.

The inscribed circle of a triangle is the largest circle contained in the triangle.

Suppose a right triangle has circumscribed circle of diameter D, and inscribed circle of diameter d.



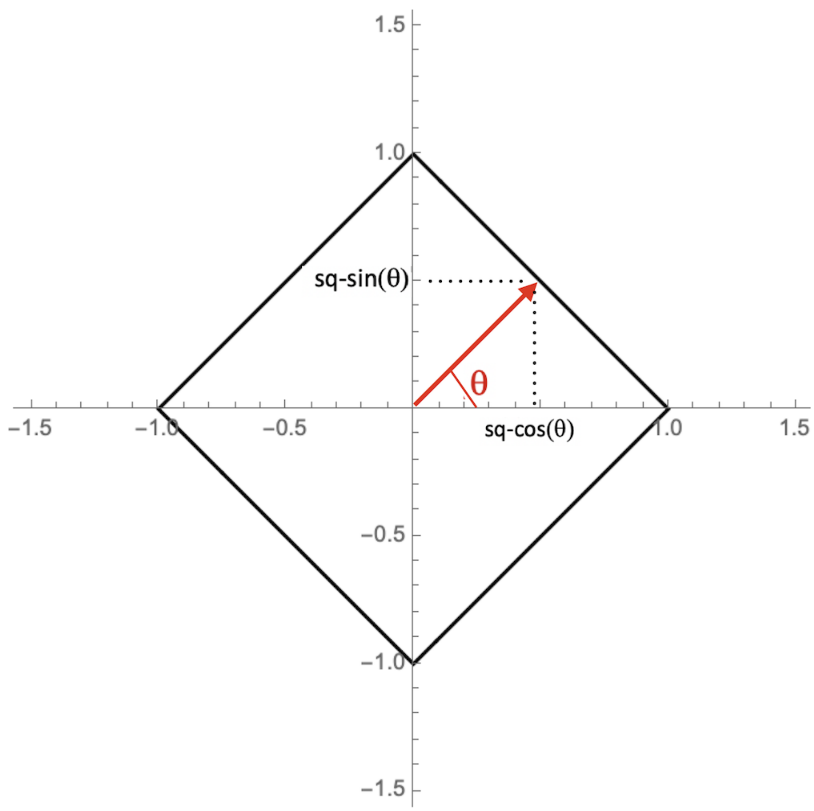
- Show that $d + D = a + b$ where a, b are the lengths of the non-hypotenuse sides of the triangle.
- Does the previous property characterize right triangles? That is, if a triangle has $d + D = a + b$ for two of the side lengths a, b is the triangle necessarily a right triangle (with hypotenuse $c = \sqrt{a^2 + b^2}$), or could there be a non-right triangle with this property?

Thanks to James Tanton for this problem.

2.34 Squiggly sine and squiggly cosine

Shown in the picture below is a plot of

$$\{(x, y) : |x| + |y| = 1\}$$



As the red radius vector travels from the horizontal axis anti-clockwise around the closed loop, we define the squiggly sin, $sqsin(\theta)$, of the angle θ the radius vector makes with the horizontal axis to be the projection of the radius vector onto the vertical axis. Similarly, we define the squiggly cosine, $sqcos(\theta)$, to be the projection onto the horizontal axis.

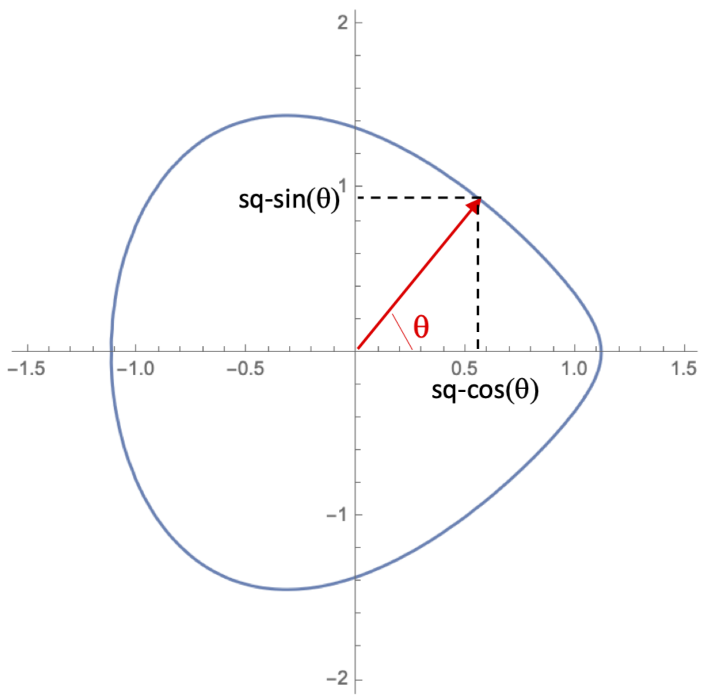
The functions $sqsin(\theta)$ and $sqcos(\theta)$ are clearly periodic: what do their graphs look like?

Is there an algebraic formula connecting $sqsin(\theta)$ and $sqcos(\theta)$?

Let's define the squiggly tangent to be $sqtan(\theta) = sqsin(\theta)/sqcos(\theta)$ when $sqcos(\theta) \neq 0$.

What does the graph of $sqtan(\theta)$ look like?

Try this again, this time replacing $\{(x, y) : |x| + |y| = 1\}$ with the closed loop in the graph of $y^2 = x^3 - 3x^2/2 - 5x/4 + 15/8$:



Thanks to James Tanton for the germ of this idea.

3 Investigations

3.1 Polynomials with odd integer coefficients and integer roots

There is no polynomial $p(x) := ax^2 + bx + c$ of degree 2 in which the coefficients a, b, c are all odd integers, and p crosses the x-axis at 2 distinct integer points.

Why is that?

However there *is* a polynomial $p(x) := ax^3 + bx^2 + cx + d$ of degree 3 in which the coefficients a, b, c, d are all odd integers, and p crosses the x-axis at 3 distinct integer points. For example,

$$p(x) := x^3 - 9x^2 + 23x - 15$$

crosses the x-axis at $x = 1, 3, 5$.

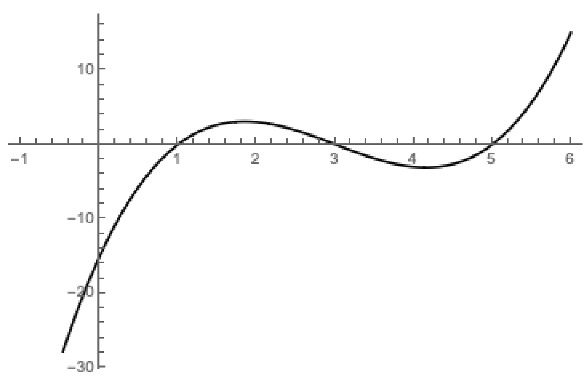


Figure 11: Plot of $x^3 - 9x^2 + 23x - 15$

Investigate for which positive integers n there is a polynomial

$$p(x) := a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

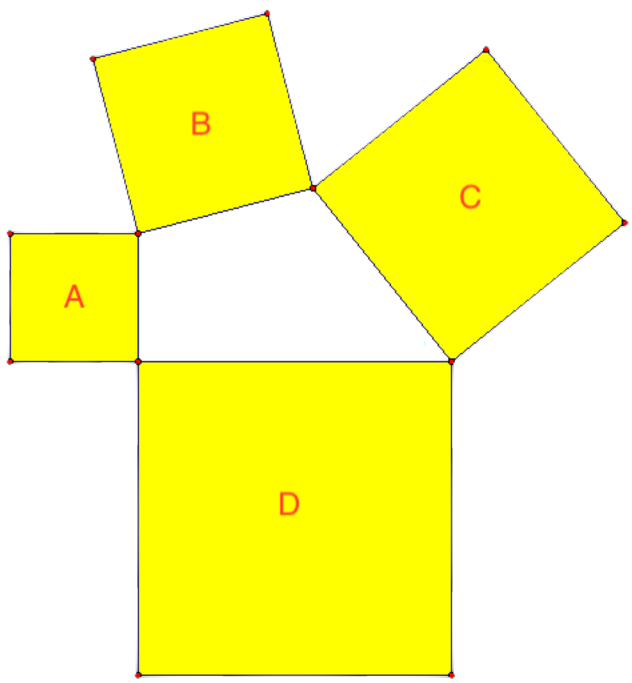
of degree n in which the coefficients $a_n, a_{n-1}, \ldots, a_1, a_0$ are all odd integers, and p crosses the x-axis at n distinct integer points.

Thanks to James Tanton for this problem.

3.2 Area of squares on the side of a quadrilateral

For the squares on the sides of a right triangle of areas A, B, C (non-decreasing magnitude)we have A+B = C (Pythagorean theorem)

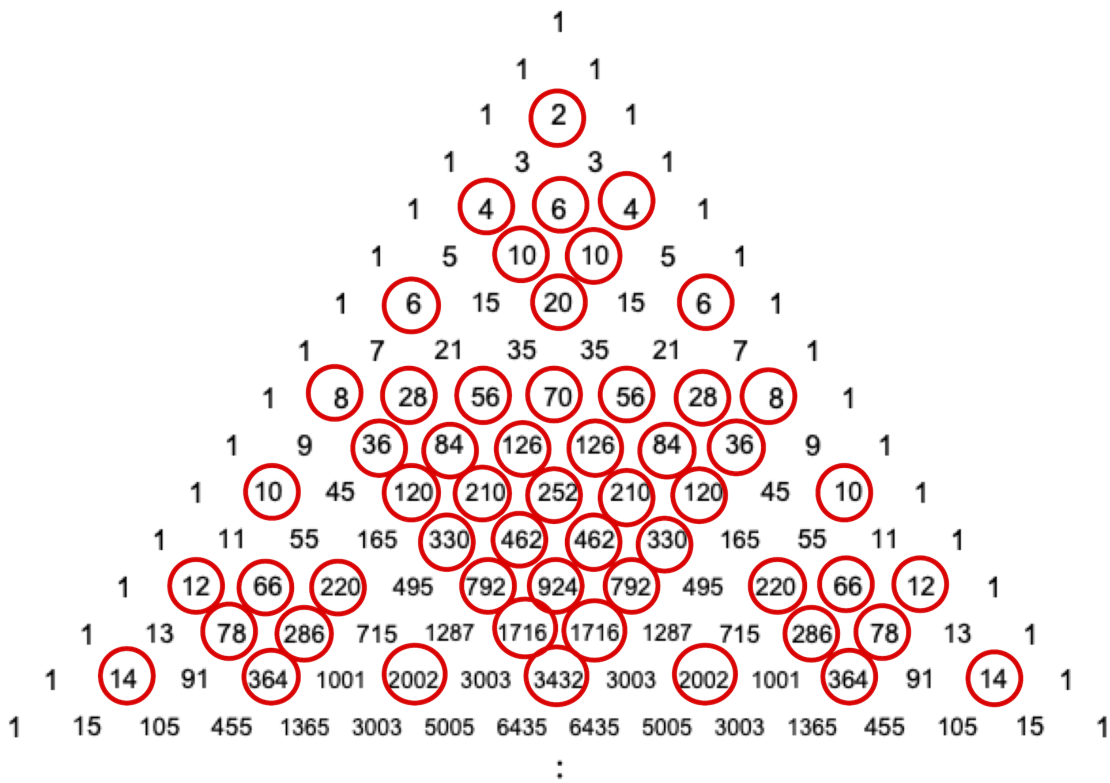
Investigate quadrilaterals such that for the squares on its sides, with areas A, B, C, D (non-decreasing) we have A+B+C=D



Thanks to James Tanton for this problem and Dean Ballard for the quadrilateral picture.

3.3 Even numbers in Pascal's triangle

As we move down the rows of Pascal's triangle, *even numbers* seem to appear in a not entirely predictable way:



The table below shows the proportion of even numbers up to and including row n for $n = 0, \ldots, 15$:

Row #	# evens	Proportion evens
0	0	0
1	0	0
2	1	$\frac{1}{6}$
3	1	$\frac{1}{10}$
4	4	$\frac{4}{15}$
5	6	$\frac{2}{7}$
6	9	$\frac{9}{28}$
7	9	$\frac{1}{4}$
8	16	$\frac{16}{45}$
9	22	$\frac{2}{5}$
10	29	$\frac{29}{66}$
11	33	$\frac{11}{26}$
12	42	$\frac{6}{13}$
13	48	$\frac{16}{35}$
14	55	$\frac{11}{24}$
15	55	$\frac{55}{136}$

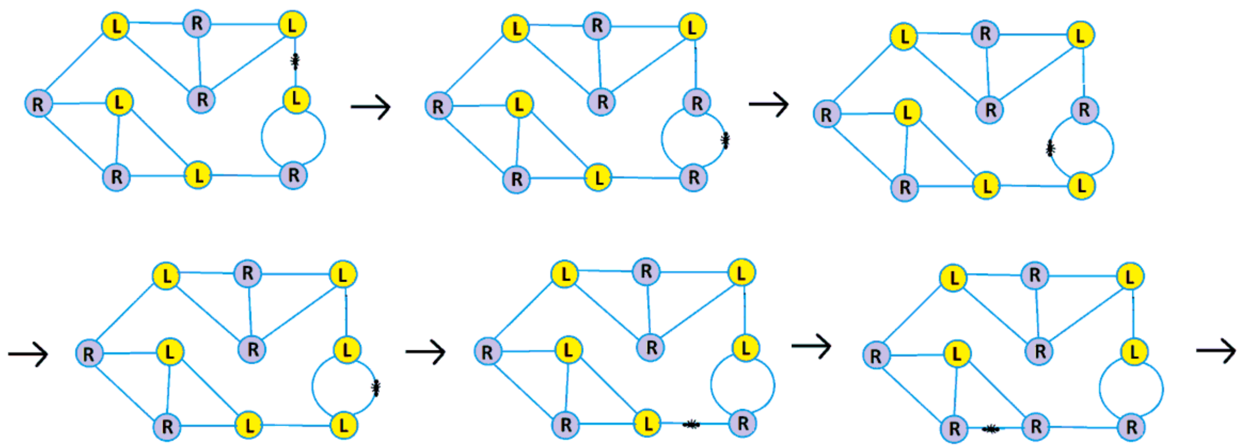
Investigate the pattern of occurrences of even numbers as you move down the rows of Pascal's triangle.

Investigate if and how the proportion of even numbers up to and including row n behaves as a function of n. Does this proportion approach a limiting value as n increases?

Thanks to Matt Enlow for this problem

3.4 Ant walks

An ant walks along the trail system shown:



When the ant gets to an L node, it turns left and changes L to an R. When the ant gets to an R node, it turns right and changes R to an L.

Does the ant visit each node?

For which initial pattern of Ls & Rs does the ant *not* visit each node?

The trail system shown us an example of a labeled connected graph: the graph is *connected* because it is possible to travel form any node to any other node along a path of edges; it is *labeled* because each node has either a “L” label or a “R” label. Explore which well known connected graphs can be labeled with Ls and Rs so that the ant can visit each node of the graph, turning left at L nodes and changing them to R, and vice versa with R nodes.

Thanks to James Tanton for this problem, who created it in honor of Christopher Langton.

3.5 Number of digits in powers of 2

The number of digits in 2^n for $1 \leq n \leq 30$ is shown in the table below:

n	2^n	Number of digits
1	2	1
2	4	1
3	8	1
4	16	2
5	32	2
6	64	2
7	128	3
8	256	3
9	512	3
10	1024	4
11	2048	4
12	4096	4
13	8192	4
14	16 384	5
15	32 768	5
16	65 536	5
17	131 072	6
18	262 144	6
19	524 288	6
20	1 048 576	7
21	2 097 152	7
22	4 194 304	7
23	8 388 608	7
24	16 777 216	8
25	33 554 432	8
26	67 108 864	8
27	134 217 728	9
28	268 435 456	9
29	536 870 912	9
30	1 073 741 824	10

You can see there are 3 occurrences of powers of 2 with 1 digits, 3 occurrences with 2 digits, 3 occurrences with 3 digits and 4 occurrences with 4 digits.

Also, each number of digits seems to occur only either 3 or 4 times:

Does a given number of digits only occur either 3 or 4 times?

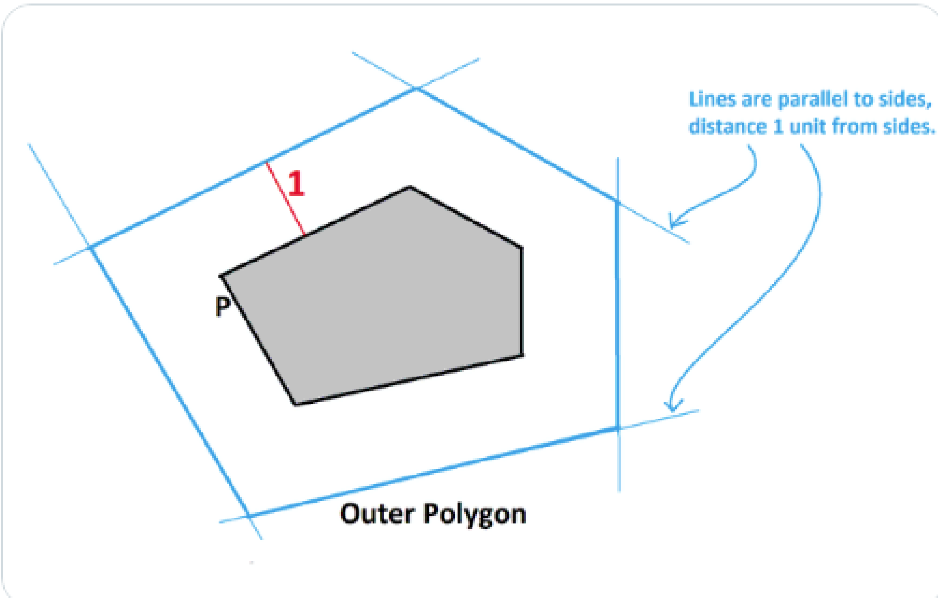
Number of digits	Number of occurences
1	3
2	3
3	3
4	4
5	3
6	3
7	4
8	3
9	3
10	4
11	3
12	3
13	4
14	3
15	3
16	4
17	3
18	3
19	4
20	3
21	3
22	4
23	3
24	3
25	4
26	3
27	3
28	4
29	3
30	3

Do the 3's and 4's go on alternating, or is it all 3's or all 4's from some point on?

Is there some discernible pattern to the occurrences of 3's and 4's?

3.6 Similarity of an outer polygon?

For a convex polygon P, produce line segments 1 unit outside each edge of the polygon to create a new polygon Q:

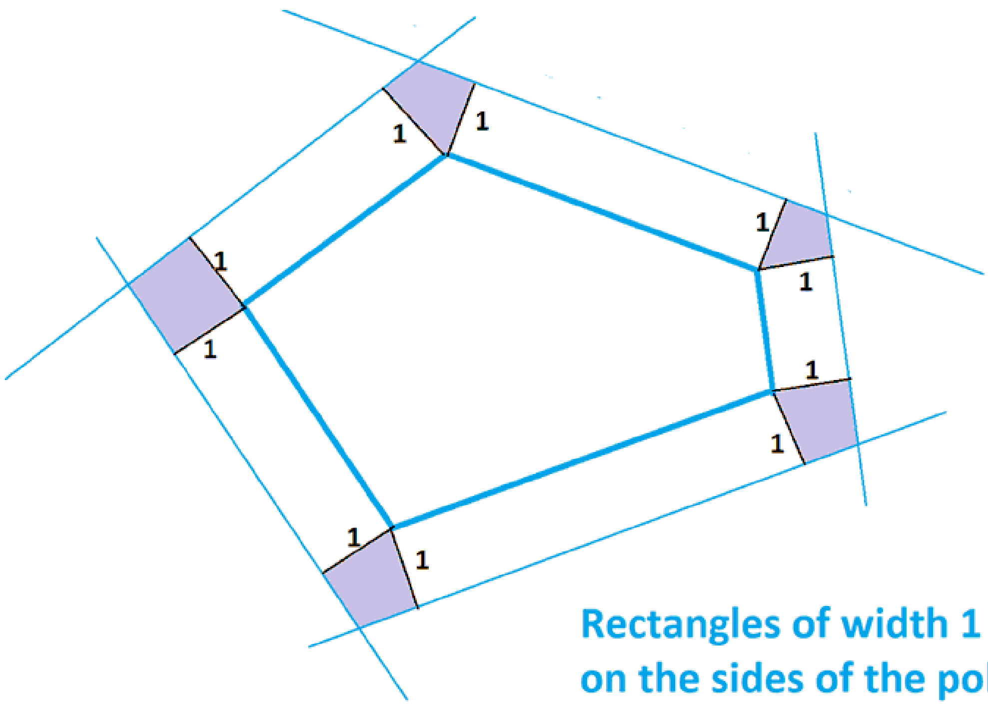


When is this outer polygon Q similar to the polygon P?

Thanks to James Tanton for this problem.

3.7 Polygonal problems

Define Π for a convex polygon to be sum of the areas of the shaded regions shown in the figure below:



Rectangles of width 1 are constructed on the sides of the polygon.

What is the value of Π for a regular n-gon?

As n increases, dos Π converge to a limit?

What if the polygon is is not regular?

Suppose a convex polygon has area A, perimeter P, and Π value as shown above.

If the polygon circumscribes a circle of radius 1, show that:

- $A = \Pi$
- $P = 2\Pi$

Is the converse true: If $A = \Pi$ and $P = 2\Pi$ must the polygon circumscribe a circle of radius 1?

Thanks to James Tanton for these problems.

3.8 π-values and the Koch snowflake

For a polygon P James Tanton defines a “pi-value” $\pi(P)$ as

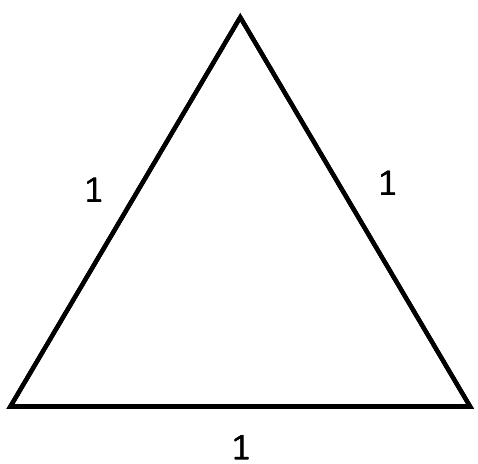
$$\pi(P) := \frac{L^2}{4A}$$

where L is the sum of the lengths of the edges of P (the perimeter of P) and A is the area of P.

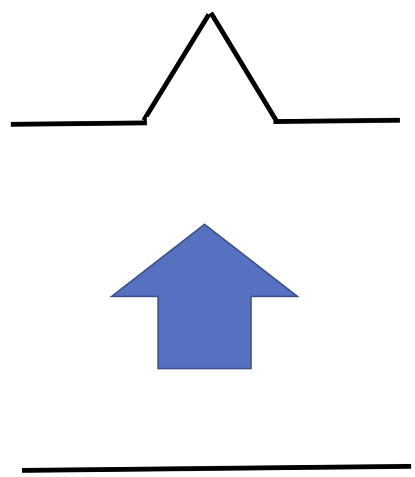
In the video referenced he defines pi-values for more general figures (technically, those bounded regions with a well-defined area, whose boundary is a simple closed curve with a well-defined length) but we will only use his pi-values idea here for polygons.

The Koch Snowflake K is constructed as the limit of a sequence of polygonal perimeters K_n:

K_1 is an equilateral triangle with side lengths 1:



K_{n+1} is obtained from K_n by replacing each straight line segment in the perimeter of K_n by a "bent" piece as follows:



Each line segment is divided into thirds and the two legs of a smaller equilateral triangle are constructed on the, then deleted, middle third of the original line segment.

The first 4 stages, K_1 through K_4, of the construction of the Koch snowflake are shown below:

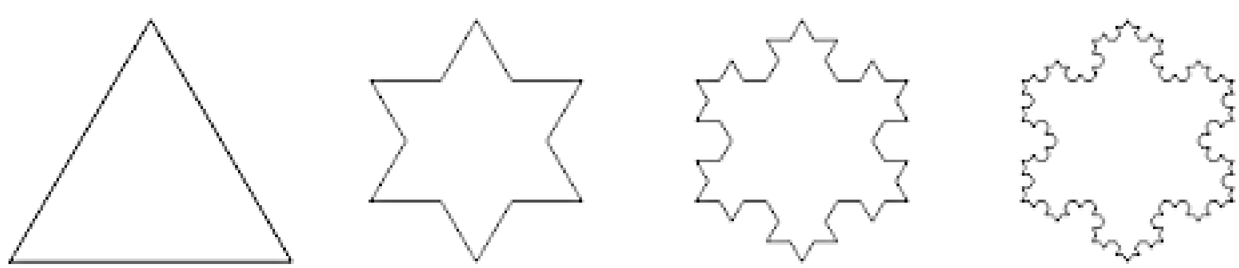
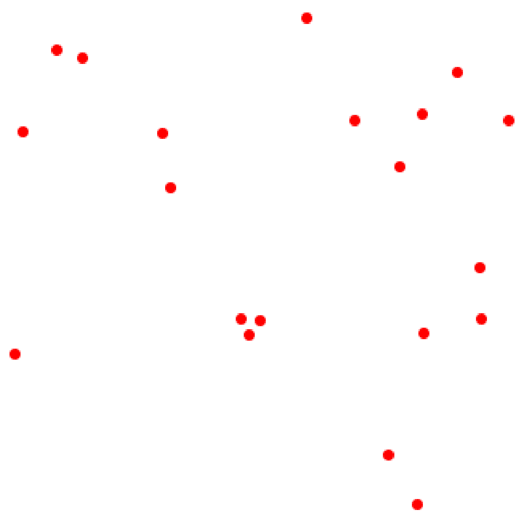


Figure 12: K_1 through K_4

Investigate the pi-values $\pi(K_n)$ and how they vary with n.

3.9 An algorithm for a printing path

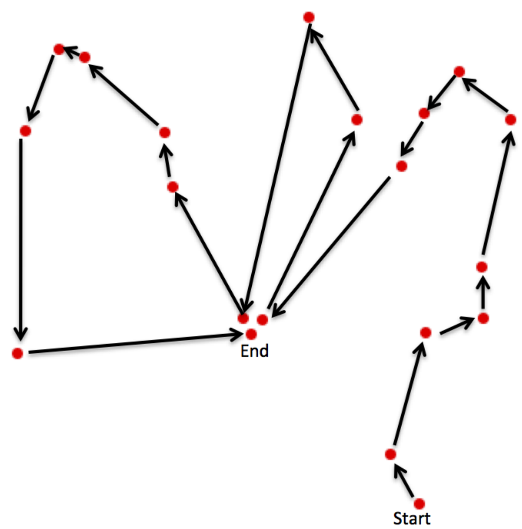
Suppose we have a collection of points in a plane, for example (but the points might also be in 3-dimensional space):



A *printing path* on the collection of points consists of:

1. a point designated as "start"
2. a point designated as "end"
3. for each point other than the start and end, an arrow into that point from another point, and an arrow out of that point to another point
4. a arrow out of the start point to another point
5. an arrow into the end point from another point
6. the arrows do not cross or meet except at the specified points

In the language of directed graphs, a *printing path* on the points is a plane directed graph with the points as vertices in which each vertex has in-degree 1 and out-degree 1, except for the start point which has in-degree 0 and out-degree 1, and the end point which has in-degree 1 and out-degree 0.



Can you devise an algorithm that, given the points, constructs a printing path on the points? Is this always possible? Can you devise an algorithm so that the total length of the arrows joining the points is as small as possible?

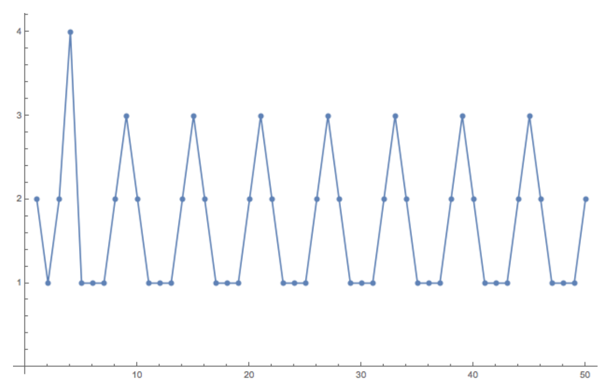
Thanks to Alfa Heryudono for this problem.

3.10 Investigating Mahler's 3/2 problem

Kurt Mahler was a German born mathematician who, among other appointments, spent many years as Professor of Mathematics at the Institute for Advanced Studies, the Australian National University. Mahler conjectured that for every real number x there is a positive integer n such that the fractional part of $x(\frac{3}{2})^n$ is $\frac{1}{2}$ or greater. This problem is still unsolved.

Investigate for a specific real number - for example $x = \frac{1+\sqrt{5}}{2}$ - and for each natural number n, what is the least positive integer k_n for which the fractional part of $x^n(\frac{3}{2})^{k_n} \geq \frac{1}{2}$. In other words, we are assuming that for *each* of $x, x^2, x^3, x^4, \ldots$ there will be a positive integer k_n for which the fractional part of $x^n(\frac{3}{2})^{k_n} \geq \frac{1}{2}$, and we want to compute the first such k_n given a power x^n of x.

Is there some pattern? Can you make sense of, or explain, any patterns you see? For example, for $x = \frac{1+\sqrt{5}}{2}$, below is a plot for each n (horizontal axis) of the first k_n (vertical axis) for which the fractional part of $x^n(\frac{3}{2})^{k_n} \geq \frac{1}{2}$:



3.11 Primes between successive Fibonacci numbers

The n^{th} Fibonacci number, denoted $F(n)$ is defined recursively as:

- $F(0) = 0$
- $F(1) = 1$
- $F(n) = F(n-1) + F(n-2)$ for $n \geq 2$

Investigate how the number of primes between $F(n)$ and $F(n+1)$, inclusive, grows with n.

As a variant on this problem, Investigate how the number of primes between the n^{th} and $(n+1)^{st}$, inclusive, Catalan numbers grows with n.

3.12 Numbers with a factor having the same number of 1s, base 2

James Tanton (@jamestanton) asked the following question on Twitter, Wednesday, August 8, 2018:

"Which positive integers n have a factor $k < n$ so that n and k have the same number of 1s in binary?"

Is it obvious this is true for powers of 2?

What about prime numbers n?

And what about squares of primes?

As a variant, which positive integers n have a factor $k < n$ so that n and k have the same number of 0s in binary?

3.13 Occurrences of the digit 2 in the base 3 expansion of powers of 2

The base 3 expansion of $2^8 = 256$ is 100111, because

$$1 \times 3^5 + 0 \times 3^4 + 0 \times 3^3 + 1 \times 3^2 + 1 \times 3^1 + 1 \times 3^0 = 256$$

There is no digit "2" in the base 3 expansion of $2^8 = 256$, and a famous conjecture of Paul Erdös is that this is the last n for which 2^n has no digits 2 in its base 3 expansion: namely, Erdös conjectures that for all $n > 8$, the base 3 expansion of 2^n does contain the digit 2 at least once.

Can we be more quantitative about this? What does experiment suggest is the *average number* of occurrences of the digit 2 in the base 2 expansion of 2^n?

In other words, suppose we compute the number of occurrences of the digit 2 in the base 3 expansion of 2^k for all $k \leq n$ and form the average of all those numbers. What is a good estimate of how that average varies with n?

3.14 Prime values of $n^p + p^n$

For a positive integer n let FTP(n) denote the first positive integer p for which $n^p + p^n$ is prime (or ∞ if there is no such p).

For example FTP(3) = 2 because $3^2 + 2^3 = 17$ is prime, but $3^p + p^3$ is not prime for $p = 1$.

Similarly, FTP(5) = 24 because $5^{24} + 24^5 = 59604644783353249$ is prime, but $5^p + p^5$ is not prime for any $p < 24$.

A list of the values FTP(n) for n from 1 through 7 is:

n	FTP(n)
1	1
2	1
3	2
4	1
5	24
6	1
7	54

Investigate further values of FTP(n): in particular, what are the values FTP(8) through FTP(13)?

3.15 Binary disjoints

For a positive integer n, call a positive integer $k < n$ a *binary disjoint* of n if the binary representations of k and n have no 1's in common places.

For example, 5 is a binary disjoint of 10 because $5 = 101$ base 2, and $10 = 1010$ base 2: 5 has 1's in the 2^0 and 2^2 places, while 10 has 1's in the 2^1 and 2^3 places.

10 is the sum of its binary disjoints: 1, 4, 5. Is there any other positive integer that is the sum of its binary disjoints?

James Tanton (@jamestanton) asked on Twitter on April 14, 2018, which n are a multiple of each of their binary disjoints - are these just the numbers of the form $n = 2^k - 2$, or are there other such numbers?

Which n have only 1 as a binary disjoint?

Note that 9 and 25 have the same set of binary disjoints: 2, 4, 6. Let's call such a pair "binary disjoint friends". What other pairs of binary disjoint friends can you find?

3.16 Probability random points are in convex position

Suppose that n points are chosen uniformly randomly and independently from inside the square $[0, 1] \times [0, 1]$.

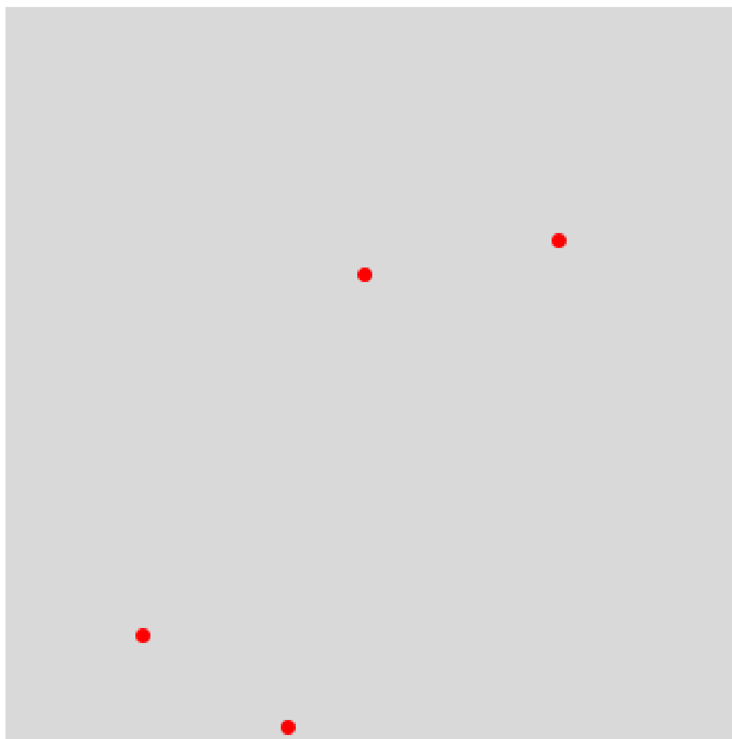


Figure 13: 4 uniformly random points in a square

The points are in *convex position* if each point is an extreme point of the convex hull of all the points.

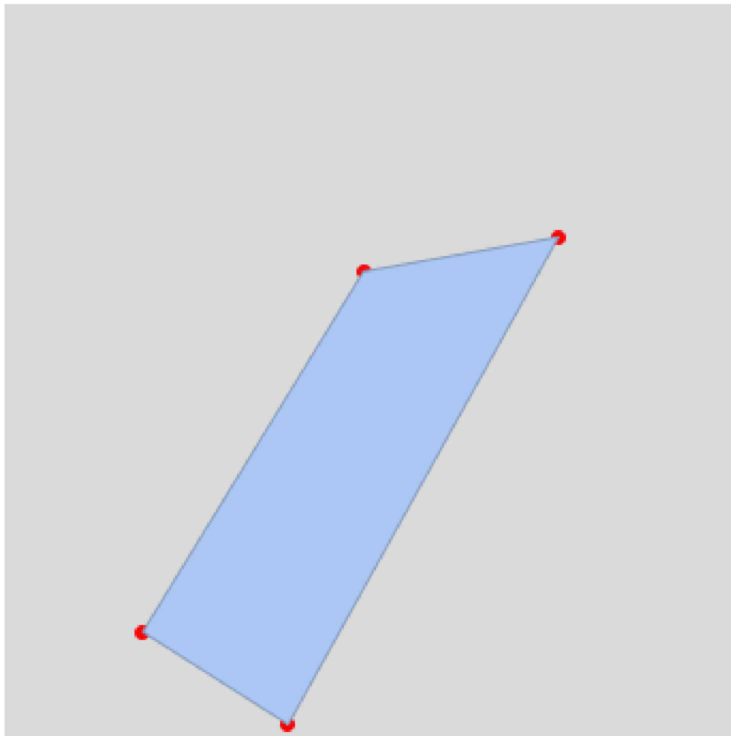


Figure 14: Convex hull of 4 uniformly random points in a square

What is the probability, as a function of n, that n uniformly random points in the square are in convex position ?

3.17 Numerators of the fractional parts of powers of 3/2

It is a famous long-standing problem whether the fractional parts of $(\frac{3}{2})^n$ are uniformly distributed in the interval $[0, 1]$.

If we denote by $\text{fp}(x)$ the fractional part of a real number x, "uniformly distributed" means for all $0 \le a < b \le 1$

$$\frac{\#\{k \le n : a \le \text{fp}((3/2)^k) \le b\}}{n} \to b - a \text{ as } n \to \infty$$

This is a notoriously difficult problem on which mathematicians are actively working.

What, however can you say about the behavior, or even the *average behavior*, of the numerators of the fractional parts of $(\frac{3}{2})^n$? The first 20 numerators are:

1, 1, 3, 1, 19, 25, 11, 161, 227, 681, 1019, 3057, 5075, 15225, 29291, 55105, 34243, 233801, 439259, 269201

3.18 Distribution of areas of right triangles with rational sides

Imagine going down the Calkin-Wilf tree (see the section "The Calkin-Wilf tree") to, for example, the 10^{th} level.

Form all pairs of rational numbers $(\frac{a}{b}, \frac{c}{d})$ and select those pairs for which $\frac{a}{b} < \frac{c}{d}$ and $(\frac{a}{b})^2 + (\frac{c}{d})^2$ is a rational number.

Such pairs give a right triangle with rational side lengths $\frac{a}{b}, \frac{c}{d}$ and

$$\frac{e}{f} = \sqrt{(\frac{a}{b})^2 + (\frac{c}{d})^2}$$

with rational area $\frac{a/b \times c/d}{2}$.

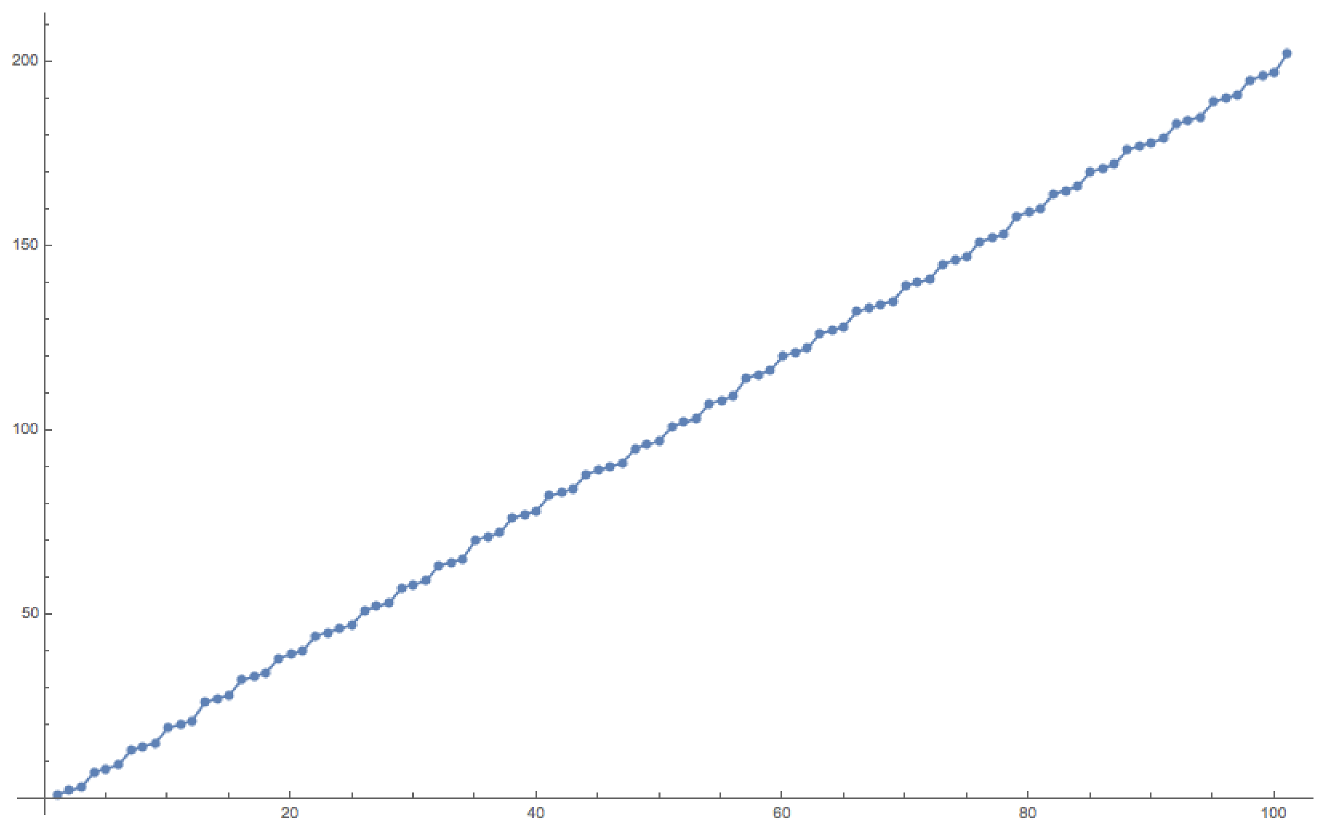
What can you say about the probability distribution of those areas?

3.19 Positive integers with positive sine

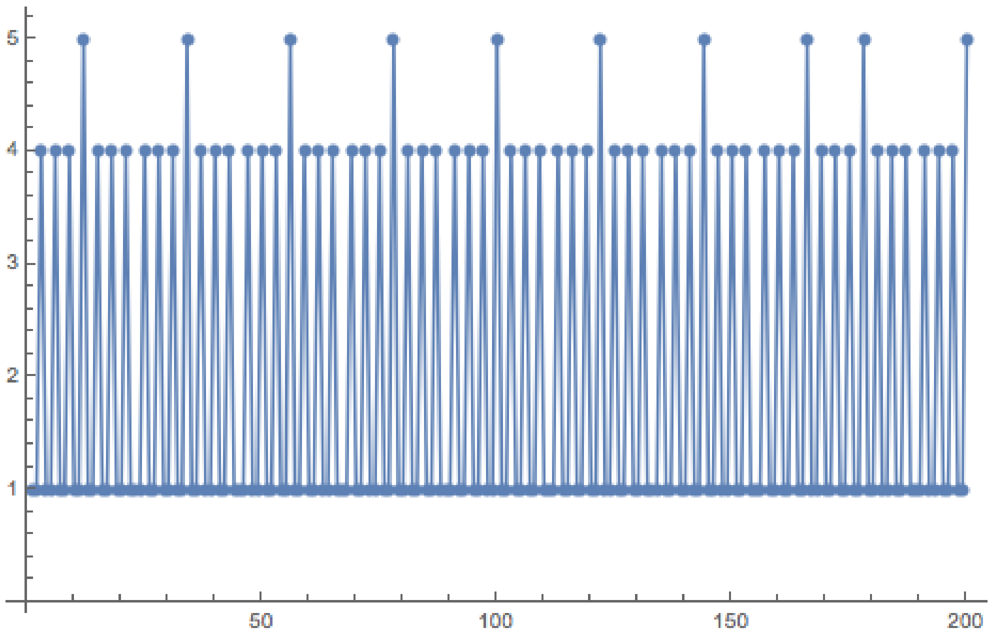
A short list of positive integers n for which $\sin(n) > 0$ appears in the Online Encyclopedia of Integer Sequences: A070752.

Can you characterize these positive integers in some other way?

Here's a a plot of the first 100 such integers:



and here's a plot of the first 200 first-differences:



3.20 Distribution of $\sin^2(n)$

We can try to assign a probability to the - potentially infinite - set of positive integers k for which $\sin^2(n) > z$ where $0 \le z \le 1$ as follows:

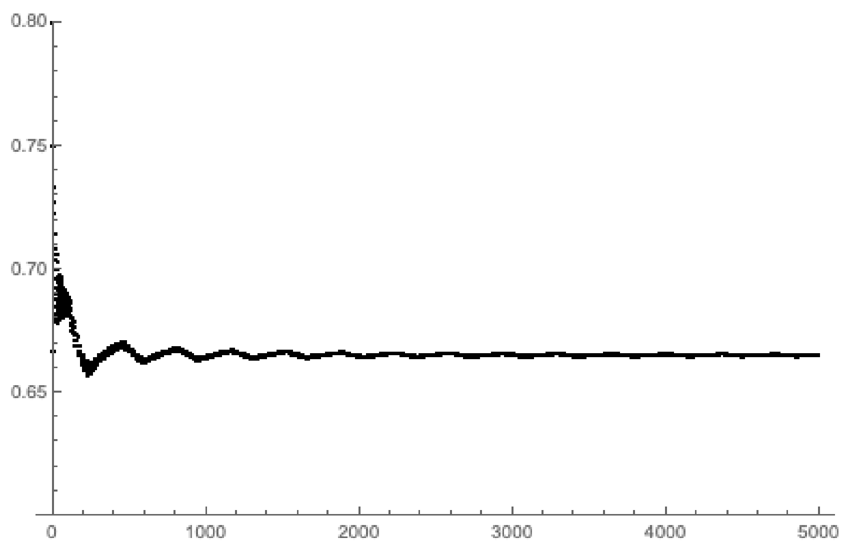
$$\mathbb{P}\mathrm{r}(\sin^2(k) > z) := \lim_{n\to\infty} \frac{\#\{k : k \le n \text{ and } \sin^2(k) > z\}}{n}$$

assuming this limit exists.

For example, for $z = \frac{1}{2}$ we plot

$$\frac{\#\{k : k \le n \text{ and } \sin^2(k) > \frac{1}{2}\}}{n}$$

versus n for n from 1 to 5000:



which appears to converge to approximately $\frac{2}{3}$.

So the first question is: does

$$\lim_{n \to \infty} \frac{\#\{k : k \leq n \text{ and } \sin^2(k) > z\}}{n}$$

exist for all $0 \leq z \leq 1$?

Investigate this question computationally and theoretically.

Then, what does the distribution of $\sin^2(n)$ for n a positive integer look like? Compute many values - say for n from 1 to $50,000$ - and describe a histogram of those values.

3.21 Multiplicative cost of a polynomial

In the article:

> Norfolk, M. (2021). The Cost of a Positive integer. *Rose-Hulman Undergraduate Mathematics Journal*, 22(1), 9.

the author, Maxwell Norfolk, discusses various "cost" functions for natural numbers.

In particular he discusses what we will call the "multiplicative cost" $C(n)$ of a positive integer n:

$$C(n) := \min\{n, \{C(a) + C(b) : a \times b = n\}\}$$

Generalize the multiplicative cost function to the ring of polynomials $\mathbb{Z}[x]$ with integer coefficients, where we place an order on polynomials as follows:

$$p(x) < q(x) \text{ if the leading coefficient of } q(x) - p(x) \text{ is positive}$$

What analogous results from Maxwell Norfolk's article carry over to the polynomial ring $\mathbb{Z}[x]$?

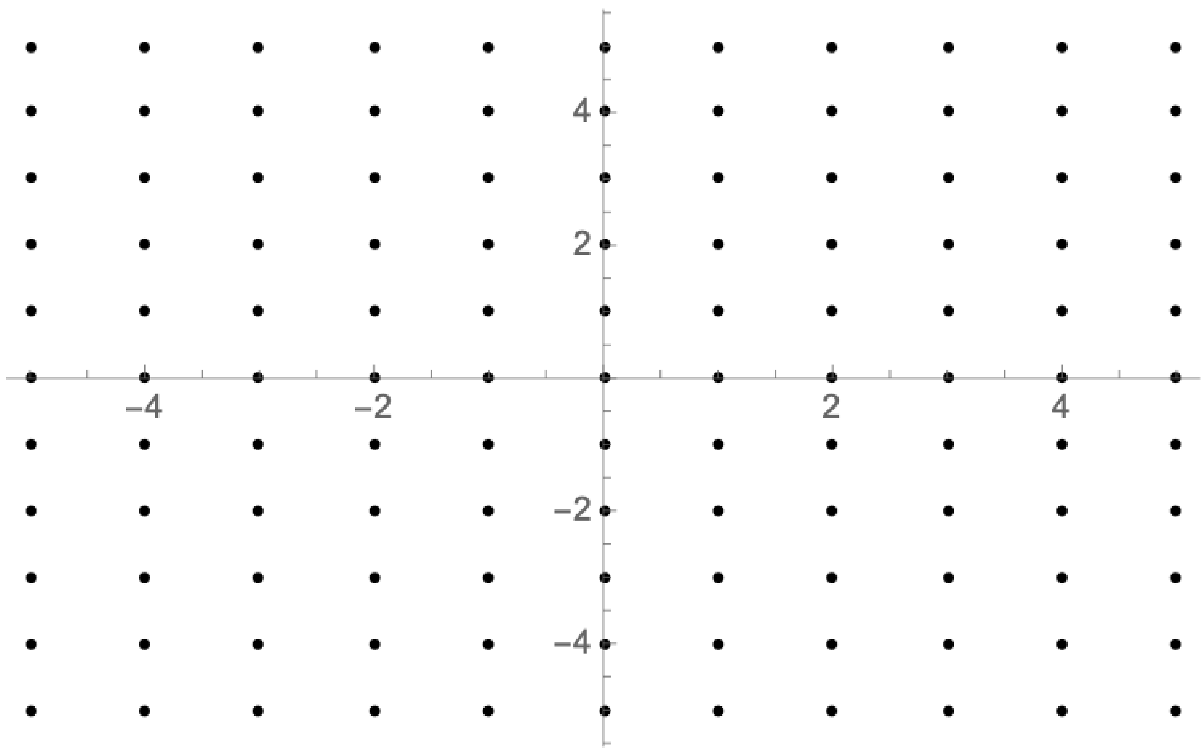
4 Research problems

The problems listed in this section are:

- Relatively easy to understand.
- Have no known published solution at the time they were written here.
- Are approachable in that a persistent and adaptable student has a chance of making good progress toward a solution.

4.1 Random walks with memory

The *integer lattice* in the plane is the set of points $p = (m, n)$ where m and n are integers (positive, negative or zero):



A random walk on the integer lattice is a finite sequence of points

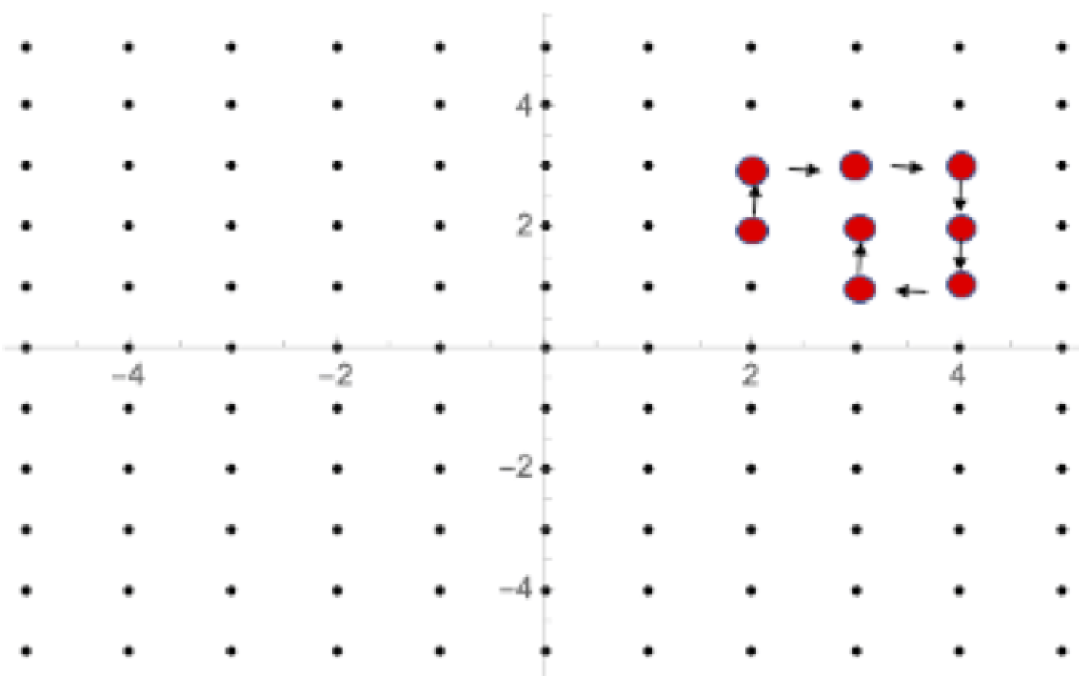
$$p_1 = (m_1, n_1), p_2 = (m_2, n_2), \ldots, p_k = (m_k, n_k)$$

where each p_{i+1} is obtained from p_i by moving up, down, left, or right one unit with probability $\frac{1}{4}$.

The *memory* of a random walk is the number of points before any given point that the random walk stores and cannot visit until they slip out of the memory store.

So, a random walk with memory 0 has no constraints – it can move up, down, left, or right with probability $\frac{1}{4}$ from any given point. A random walk with memory 1 (also called a no-backtracking random walk) is constrained to not visit the point is just previously visited. A random walk with infinite memory (also called a self-avoiding walk) gets trapped quickly, with probability 1.

A lot is known about random walks with memory $0, 1$ and ∞. However for intermediate finite memory much less is known. For example, a random walk with memory 7 can get trapped:



Questions

We consider random walks with initial point (that is, starting from) $p_1 = (0, 0)$.

1. Does a random walk of memory 7 get trapped with probability 1?
2. What is the average length of a random walk of memory 7 ?
3. Given a square of the integer lattice with corners $(\pm n, \pm n)$ what is the average number of steps for a random walk of memory 7 to exit the square?
4. What about random walks with other finite non-zero memories?
5. What about random walks with memory on other lattices – triangular or hexagonal lattices, for example?

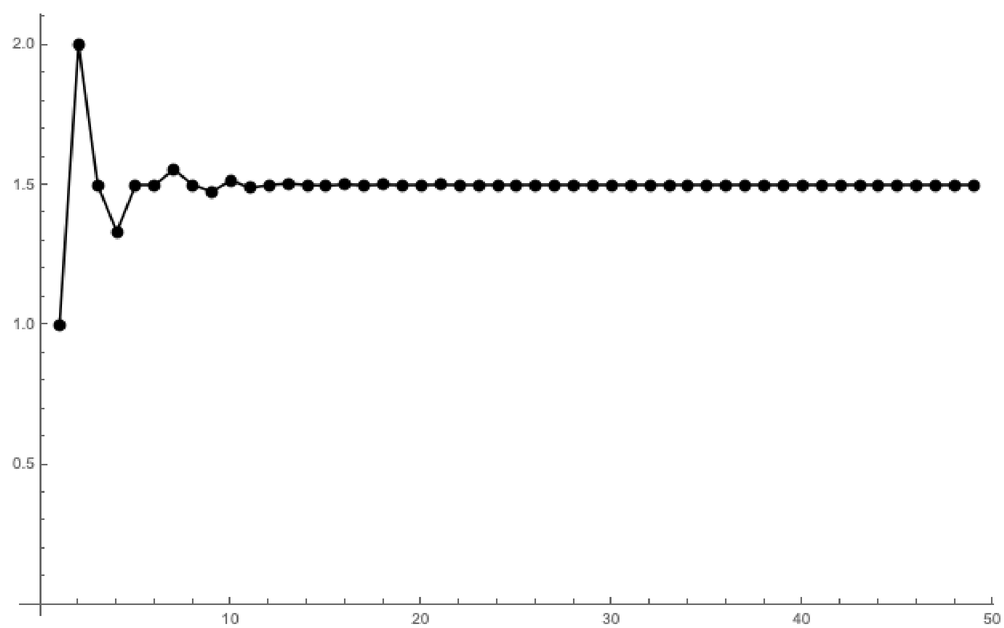
4.2 Convergence of a family of sequences?

The sequence $a(1), a(2), a(3), \ldots$ is defined recursively by:

1. $a(1) = 1$
2. $a(n) = \lceil \frac{a(1)+a(2)+\ldots+a(n-1)+1}{2} \rceil$, where for a real number x, $\lceil x \rceil$ is the *ceiling* of x - that is, the smallest integer $\geq x$.

This is sequence A005428 in the Online Encyclopedia of Integer Sequences, where you can find background on its context.

Below is a plot of the ratio $\frac{a(n+1)}{a(n)}$ of successive terms of the sequence:



It seems

$$\lim_{n\to\infty} \frac{a(n+1)}{a(n)} = \frac{3}{2}$$

Is this so?

Can you prove it?

What about sequences $a(1), a(2), a(3), \ldots$ defined recursively as:

1. $a(1) = 1$
2. $a(n) = \lceil \frac{a(1)+a(2)+\ldots+a(n-1)+p}{q} \rceil$, where p, q are positive integers.

Does the ratio $\frac{a(n+1)}{a(n)}$ approach a limit for all p, q and if so, how does the value of the limit depend on p and q?

4.3 Balanced colorings of graphs

In this problem we use the term “graph” to mean a simple graph with undirected edges as described in Wikipedia:

> A *graph* is an ordered pair $G = (V, E)$ comprising:
>
> 1. V, a set of *vertices* (also called nodes);
> 2. $E \subseteq \{\{x, y\} \mid x, y \in V \text{ and } x \neq y\}$, a set of *edges*, which are unordered pairs of vertices (that is, an edge is associated with two distinct vertices).
>
> Graphs are usually represented visually by drawing a point or circle for every vertex, and drawing a line between two vertices if they are connected by an edge.

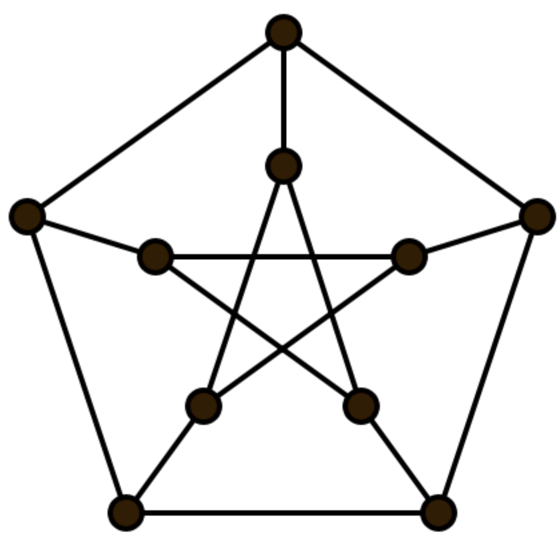


Figure 15: Drawing of a graph with 10 vertices and 15 edges

For a vertex v in a graph, a *neighbor* of v is a vertex $w \neq v$ for which there is an edge $\{v, w\}$ connecting v and w.

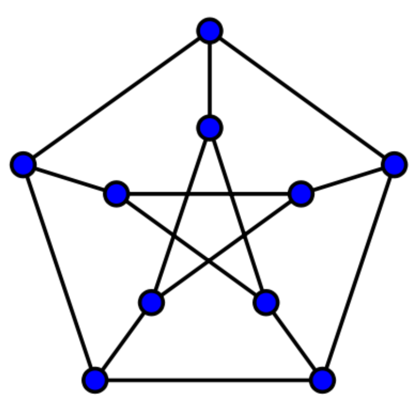
Imagine we color the vertices of a graph so that each vertex is colored with one of two colors, which we conveniently call red and blue. In the article:

> Tabatabai, P., & Gruber, D. P. (2021). Knights and liars on graphs. *Journal of Integer Sequences*, 24(2), 3.

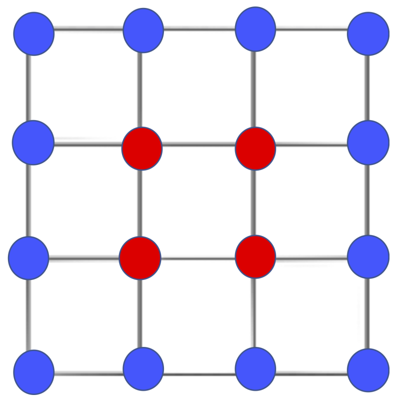
the authors call a red and blue coloring of the vertices of a graph *red-balanced* if:

- for every red vertex v it is true that exactly half the neighbors of v are red;
- for every blue vertex v it is *not true* that half the neighbors of v are red.

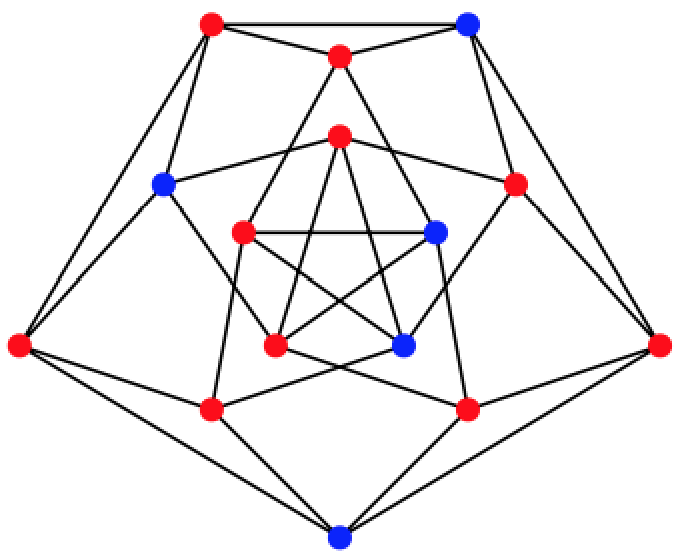
Note that the only way for the Peterson graph, drawn above, with 10 vertices and 15 edges can be colored so as to be red-balanced is for every vertex to be colored blue - that is due to the fact that every vertex has exactly 3 neighbors:



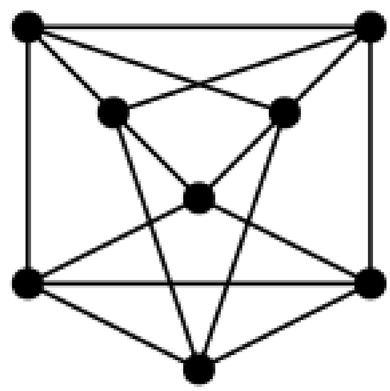
The red and blue colored graph shown below, whose vertices are on a 4×4 grid, is red-balanced:



Here is another example of a red-balanced coloring of a graph:



A graph is k-regular if all vertices have exactly k neighbors. 3-regular graphs are also called cubic graphs, and the only red-balanced coloring of a cubic graph is one in which all vertices are colored blue. Below is a 4-regular graph in which, also, the only red-balanced coloring is one in which all vertices are colored blue:



For k a positive integer, characterize $2k$-regular graphs for which the only red-balanced coloring is one in which all vertices are colored blue. In particular, which 4-regular graphs have as a red-balanced coloring only the one in which all vertices are colored blue? See here for examples of 4-regular graphs.

4.4 Cycles in the directed graphs of finite rings

In the article:

> Bounds, M. (2020). New Theorems for the Digraphs of Commutative Rings. Rose-Hulman Undergraduate Mathematics Journal, 21(1), 4.

the author, Morgan Bounds, discusses various theorems and problems related to the structure of directed graphs associated with the finite commutative ring $\mathbb{Z}_n$ of integers modulo n.

The directed graph $\Gamma(\mathbb{Z}_n)$ associated to the commutative ring $\mathbb{Z}_n$ of integers modulo n has:

1. Pairs (a, b) with $0 \leq a, b \leq n-1$ as vertices.
2. A directed edge from:

   vertex (a, b) to vertex $(a + b \pmod{n}, a \times b \pmod{n})$

An open problem in Bounds paper is to determine, from the structure of the integer n, the number and lengths of cycles in $\Gamma(\mathbb{Z}_n)$.

In another direction, we know that every finite simple ring is essentially a ring $\mathbb{M}_n(\mathbb{F}_q)$ of $n \times n$ matrices over a finite field $\mathbb{F}_q$ (Wedderburn's theorem). How do the cycles in the directed graph $\Gamma(\mathbb{M}_n(\mathbb{F}_q))$ depend on n and q?

5 Unsolved problems

Remember that as challenging and intriguing as you may find these problems they are mostly problems that several, if not many, professional mathematicians have thought about long and hard and have not yet been able to resolve. While it's sensible to be aware of these unsolved problems, and perhaps to think for a short time what the problem entails, it is generally not a wise move to spend a long time thinking about these hard unsolved problems simply because the chance of solving them is very low, and time spent on them means time taken away from other potentially solvable problems.

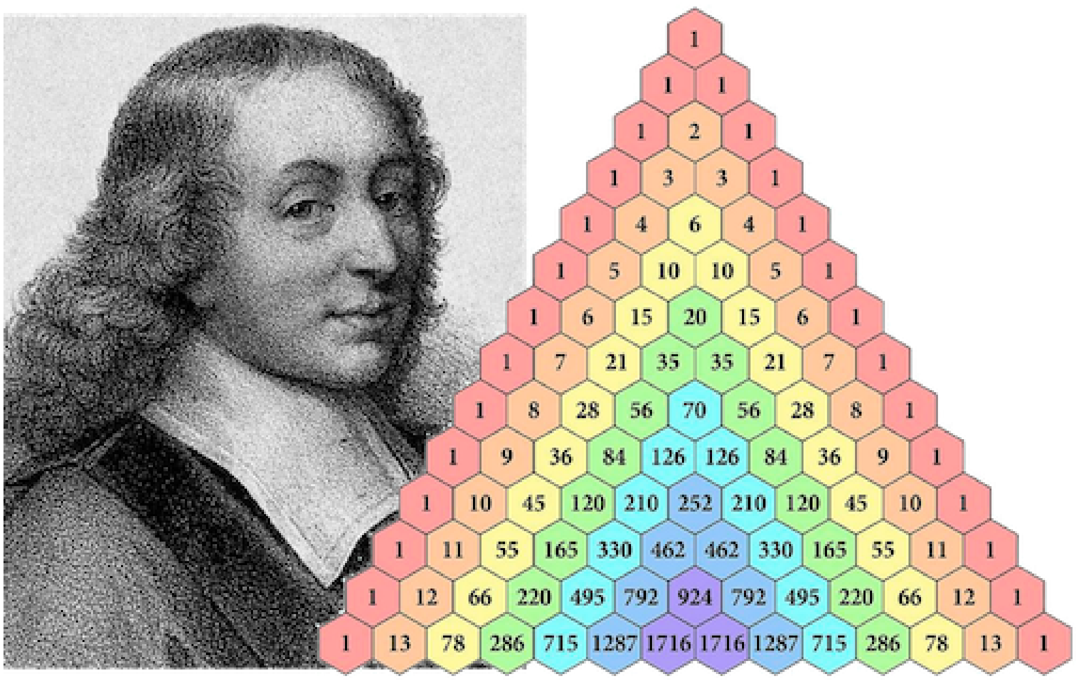
It is part of one's mathematical education to be aware of these unsolved problems - sometimes they actually get solved! - but not to become obsessed with finding a solution, because this can lead to what Richard Lipton describes as a "mathematical disease":

> **What Is a Mathematical Disease?**
>
> There is another type of "bug" that affects mathematicians—the attempt to solve certain problems. These problems have been called "diseases", which is a term coined by the great graph theorist Frank Harary. They include many famous problems from graph theory, some from algebra, some from number theory, some from complexity theory, and so on.

Have fun trying to understand what these unsolved mathematical problems are about, become more familiar with where the difficulties lie, but try to avoid becoming unhealthily obsessed with any of them. Have fun, enjoy!

5.1 How often does each positive integer occur in Pascal's triangle?

The number 1 clearly occurs infinitely often in Pascal's triangle. However integers $n > 1$ occur only a finite number of times.



Can you devise ways of calculating for any given integer $n > 1$ how often n occurs in Pascal's triangle?

A famous conjecture of David Singmaster is that there is a number N such that every integer $n > 1$ occurs no more than N times in Pascal's triangle.

So far no one has found an $n > 1$ that occurs more than 8 times in Pascal's triangle (so maybe $N = 8$?).

5.2 A prime between successive powers of an integer ?

Samantha had heard about a famous unsolved problem: that there is always a prime number between n^2 and $(n+1)^2$, for all natural numbers n.

Being a quantitative data-oriented person, Samantha did some calculations and came up with a stronger thought: "It seems to me", said Samantha, "on the basis of calculational evidence, that the number of primes between n^2 and $(n+1)^2$ is always greater than $\frac{n}{9}$".

Could Samantha be right?

What does experiment suggest is the *average number* of primes between n^2 and $(n+1)^2$? In other words, suppose we compute the number of primes between k^2 and $(k+1)^2$? for all $k \leq n$ and form the average of all those numbers. What is a good estimate of how that average varies with n?

5.3 Is there a polyomino or order 5?

A polyomino is a connected collection of squares each of which is connected to another square along an entire edge:

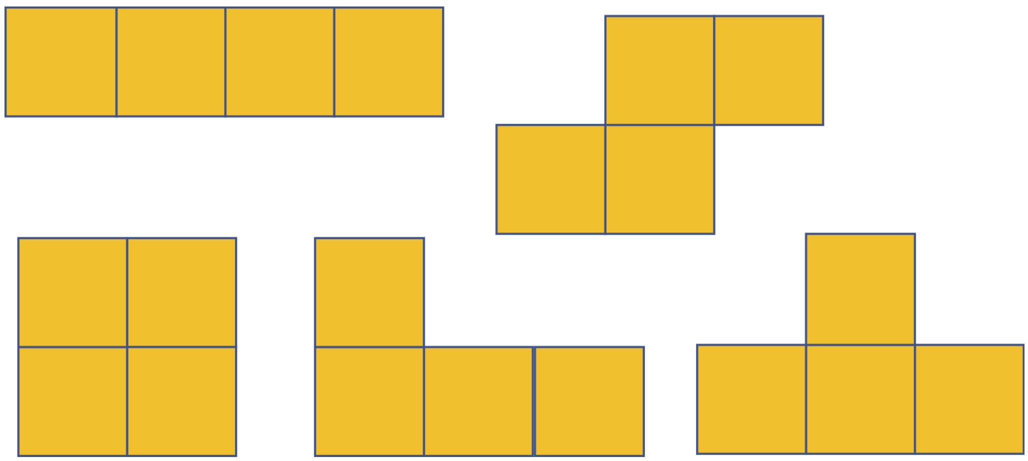


Figure 16: Polyominos constructed from 4 squares

The *order* of a polyomino is the minimum number of copies of the polyomino that can tile a rectangle (assuming that can be done).

There are polyominos of order 4:

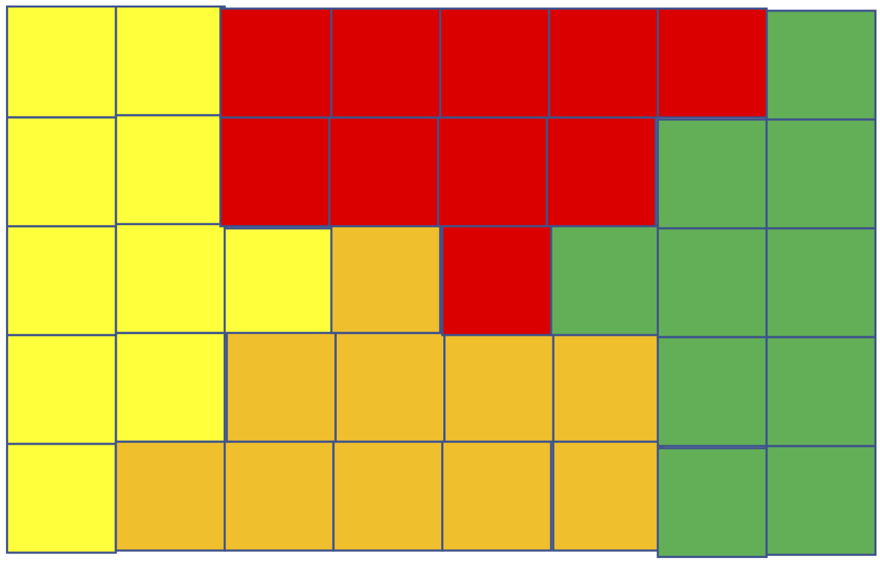


Figure 17: A rectangle tiled by 4 copies of a polyomino, no fewer copies of which tile a rectangle

There is no polyomino of order 3.

Is there a polyomino of order 5?

5.4 Runs of 0s in the binary expansion of the square root of 2

A problem of Paul Erdös asks if there are there arbitrarily long sequences of 0's in the binary expansion of $\sqrt{2}$.

5.5 Erdös–Straus conjecture

Is it true that for every positive integer n there are positive integers a, b, c such that

$$\frac{4}{n} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

See here for more details.

For recent work on this problem by a top-rate mathematician see:

> Elsholtz, C., and Tao, T. (2013). Counting the number of solutions to the Erdős–Straus equation on unit fractions. *Journal of the Australian Mathematical Society*, 94(1), 50-105.

5.6 Catalan pseudo-primes

The n^{th} Catalan number C_n is defined recursively as:

1. $C_1 = 1$
2. $C_{n+1} = \frac{2(2n+1)}{n+2} C_n$

The Catalan numbers arise in many counting problems.

Christian Aebi and Grant Cairns showed that if p is a prime number then

$$(-1)^{\frac{p-1}{2}} C_{\frac{p-1}{2}}$$

leaves a remainder of 2 when divided by p:

> Aebi, Christian, and Grant Cairns. Catalan numbers, primes, and twin primes. *Elemente der Mathematik* 63, no.4 (2008): 153-164.

They asked if this could be true for odd non-primes and found 3 such odd non-primes: 5907, 1194649, and 12327121. They call an odd composite number n a Catalan pseudo-prime if

$$(-1)^{\frac{n-1}{2}} C_{\frac{n-1}{2}}$$

leaves a remainder of 2 when divided by n:

Are there any Catalan pseudo-primes other than 5907, 1194649, and 12327121?

5.7 Cycles in cubic graphs

A connected graph is cubic if all its vertices have degree 3.

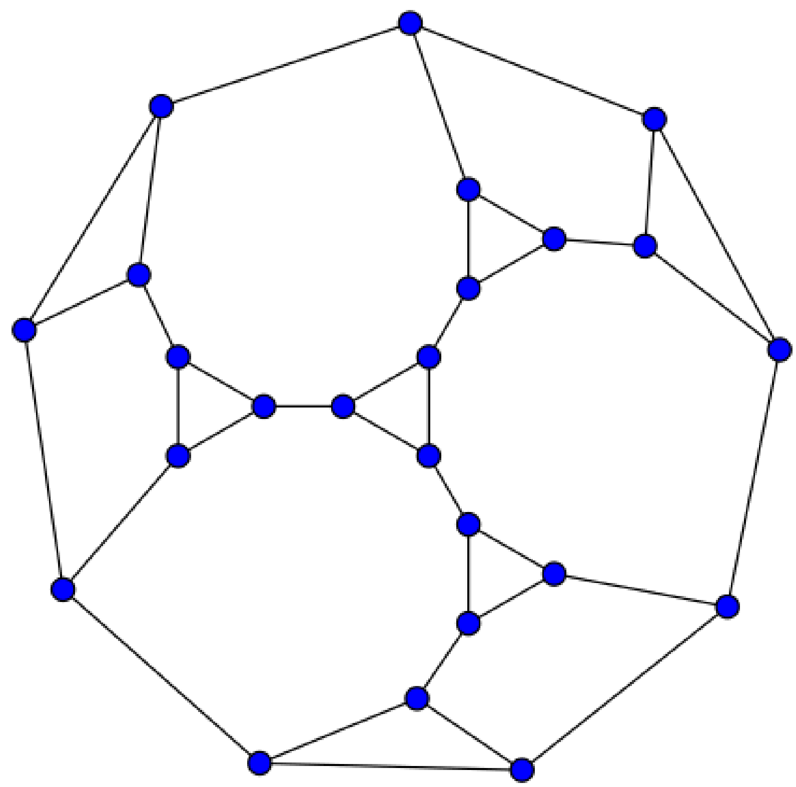


Figure 18: A cubic graph

A special case of the Erdős–Gyárfás conjecture is that every cubic graph contains a cycle of length a power of 2.

The cubic graph shown above has no cycles of length 2,4 or 8, but does have cycles of length 16:

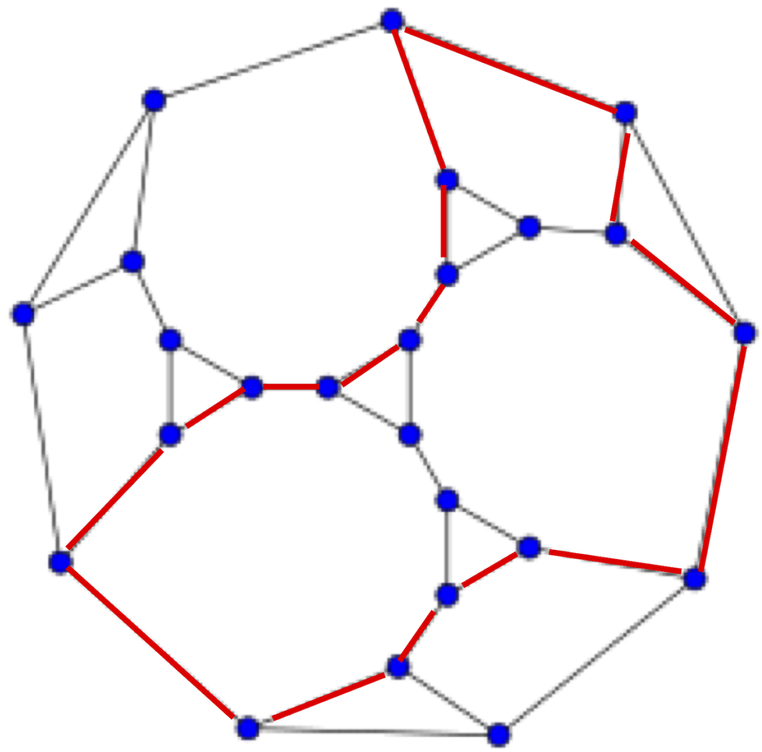


Figure 19: A cycle of length $2^4 = 16$, shown red

5.8 How many factorials modulo a prime?

For a non-negative integer n the factorial $n!$ is defined inductively as:

- $0! = 1$
- $n! = n \times (n-1)!$

It is an open problem to determine, for all prime numbers p, the size of the set

$$A(p) := \{k! (\text{mod } p) : k = 0, 1, \ldots, p-1\}$$

For example, for $p = 13$

$$A(p) = A(13) = \{1, 2, 3, 5, 6, 7, 9, 11, 12\}$$

which has size 9. Investigate how the size of $A(p)$ varies with the prime p.

Also try to estimate the *average value* of $A(p)$ - that is, for a given prime p, calculate the size of $A(k)$ for primes $k \leq p$ and average those values. Estimate how that average varies with p.

5.9 Congruent numbers

A congruent number is a positive integer n such that there is a right triangle with all sides rational numbers, and area n.

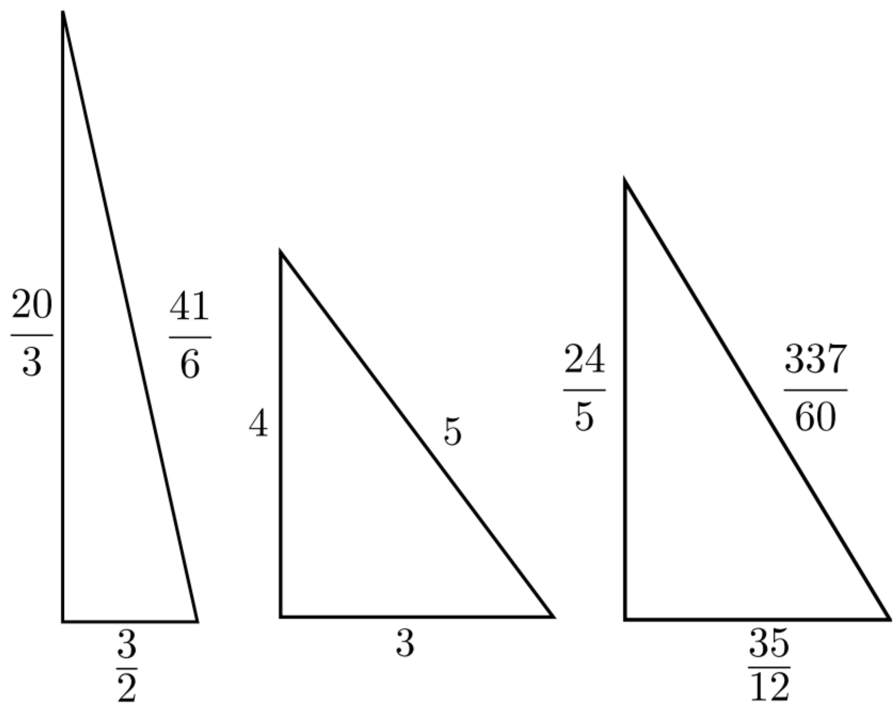


Figure 20: Rational right triangles with areas 5, 6, 7 respectively

Figure from Conrad, Keith (Fall 2008) "The congruent number problem", *Harvard College Mathematical Review*, 2 (2): 58–73

It is a major unsolved problem which positive integers are congruent numbers.

The question of whether n is a congruent number is equivalent to the question of whether the elliptic curve $y^2 = x^3 - n^2x$ has a point $(x, y), y \neq 0$, on the curve with both x, y rational numbers (a "rational point"). (Refer to the section "Features of elliptic curves" in these notes).

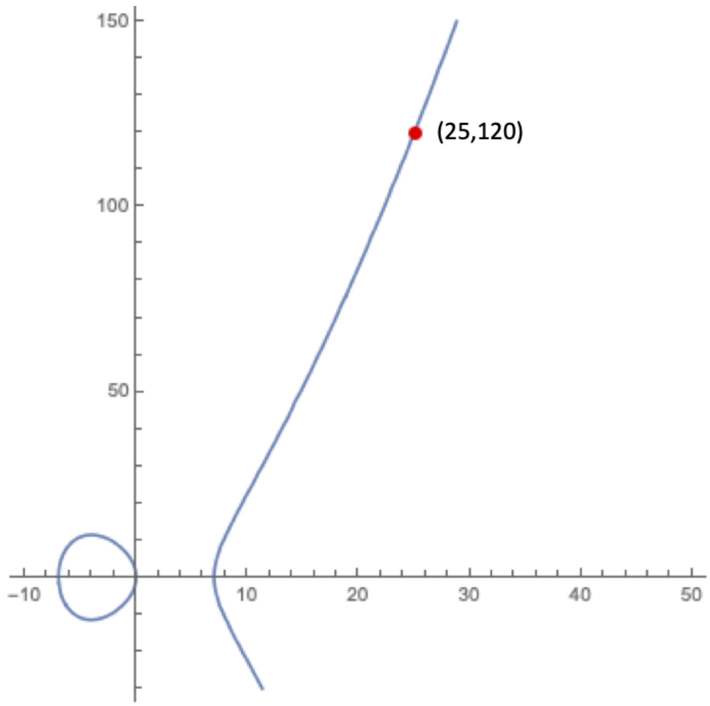


Figure 21: The rational point (25,120) on the elliptic curve $y^2 = x^3 - 7^2x$ establishes 7 as a congruent number

See Conrad, Keith (Fall 2008), "The congruent number problem", page 5, for details of a connection between rational points on elliptic curves and congruent numbers.

We could broaden our search and look for *rational numbers* $n = \frac{p}{q}$ - not simply integers - for which there is a right triangle with rational side lengths and area $n = \frac{p}{q}$.

See if, for example, you can find such a right triangle for $n = \frac{7}{2}$.

We know that $\pi \approx \frac{22}{7}$. Can you find a right triangle with rational side lengths whose area is $\frac{22}{7}$?

5.10 Small convex polygons of maximum perimeter

A convex polygon is called *small* if its diameter - the maximum distance between 2 points in the polygon - is 1.

A small square has side length $\frac{1}{\sqrt{2}}$ and therefore has perimeter $\frac{4}{\sqrt{2}} = 2\sqrt{2} \approx 2.82843$.

However there are small convex quadrilaterals with larger perimeter. The small convex quadrilateral with maximum perimeter is shown below:

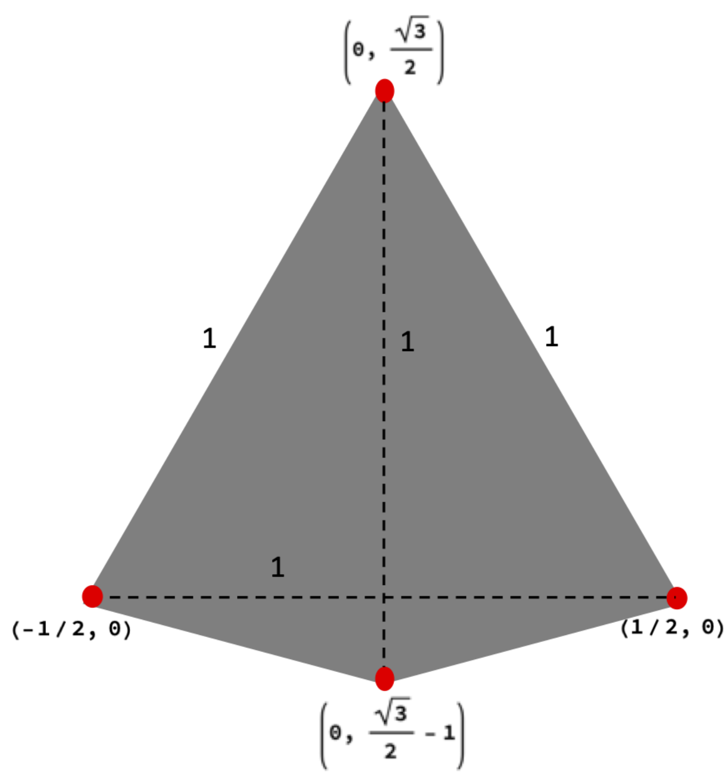


Figure 22: The convex quadrilateral with maximum perimeter $2-\sqrt{2}+\sqrt{6} \approx 3.03528$

This was established in:

> Tamvakis, N. K. (1987).On the perimeter and the area of the convex polygon of a given diameter.*Bulletin of the Hellenic Mathematical Society*, 28, 115-132.

For a given positive integer n what is a small convex polygon with n sides and maximum perimeter?

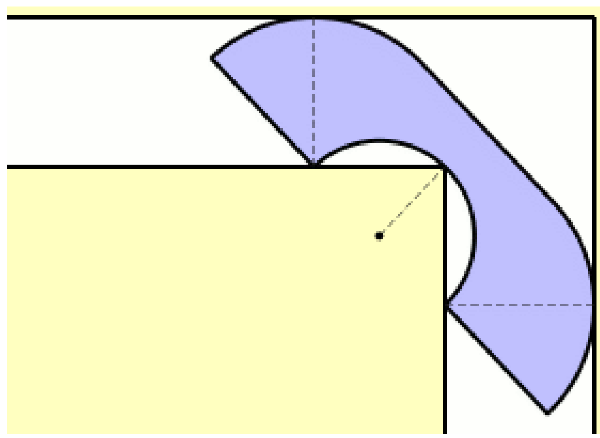
This is still an unsolved problem for general n.

For $n = 8$ - that is, small octagons - an answer, at least numerically, appears in:

> Audet, C., Hansen, P., & Messine, F. (2007). The small octagon with longest perimeter. *Journal of Combinatorial Theory*, Series A, 114(1), 135-150.

5.11 Moving a a large sofa around a corner

What is the region of largest area which can be moved around a right-angled corridor of width one?



See: "Moving Sofa Problem" at Wikipedia.

See, also, the interesting article:

> Romik, D. (2018). Differential equations and exact solutions in the moving sofa problem. *Experimental Mathematics*, 27(3), 316-330.

and:

> Kallus, Y. & Romik, D. (2018). Improved upper bounds in the moving sofa problem. *Advances in Mathematics*, 340, 960-982.

5.12 A simply stated undecidable problem?

The Collatz conjecture states that if we start from any positive integer n and repeatedly apply the function:

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ 3n+1 & \text{if } n \text{ is odd} \end{cases}$$

to get $f(n), f(f(n)), f(f(f(n))), \ldots$ we will eventually get to 1.

For example, starting from $n = 17$ and repeatedly applying the function f to the result, we reach 1 in 12 steps:

$$17 \xrightarrow{f} 52 \xrightarrow{f} 26 \xrightarrow{f} 13 \xrightarrow{f} 40 \xrightarrow{f} 20 \xrightarrow{f} 10 \xrightarrow{f} 5 \xrightarrow{f} 16 \xrightarrow{f} 8 \xrightarrow{f} 4 \xrightarrow{f} 2 \xrightarrow{f} 1$$

The Collatz conjecture has generated a huge body of work, none of it conclusive.

Patrick Honner has a nice readable article on the conjecture: "The Simple Math Problem We Still Can't Solve" in Quanta Magazine.

Terry Tao also has a very readable account of many aspects of the Collatz conjecture, which is great to see how one of the world's outstanding mathematicians thinks about such a notorious problem.

Additionally,Terry Tao's 2020 article describing his recent thinking, on and results for, the Collatz conjecture is available on the arxiv pre-print server:

> Tao, T. (2019). Almost all orbits of the Collatz map attain almost bounded values. arXiv preprint arXiv:1909.03562.

What is potentially intriguing, following a train of thought that began with John Horton Conway:

Conway, John H. (1972). Unpredictable iterations. Proc. 1972 Number Theory Conf., Univ. Colorado, Boulder. pp. 49–52.

is that this conjecture might be undecidable, largely because slightly modified problems have been proven undecidable:

Lehtonen, E. (2008). Two undecidable variants of Collatz's problems. *Theoretical Computer Science*, 407(1-3), 596-600.

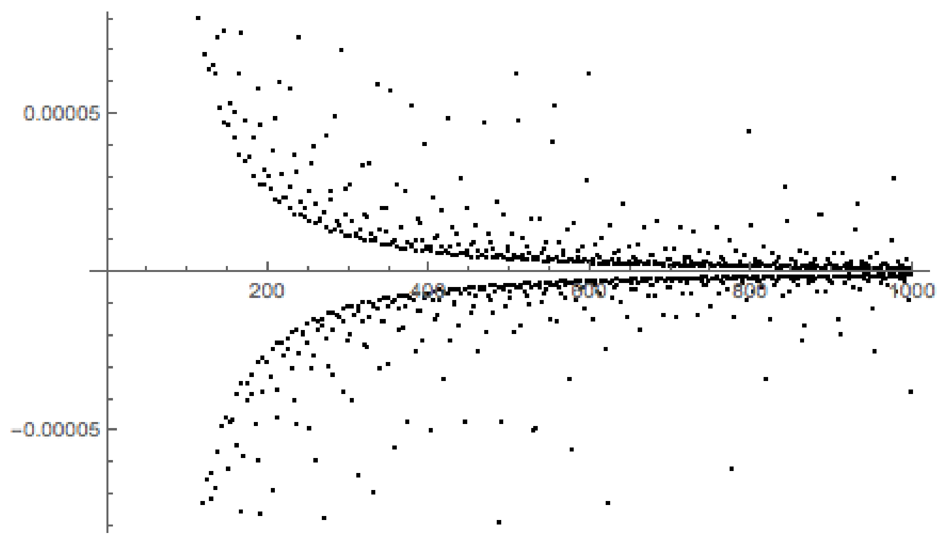
By "undecidable" we mean there is provably no algorithm to decide the conjecture in all cases.

5.13 Convergence of a sequence?

The sequence $(x_n)_{n\geq 1}$ of real numbers x_n is defined as follows:

$$x_n := \frac{1}{n^2 \sin(n)}$$

Here is (part of) a plot of the first 1000 terms $x_1, x_2, \ldots, x_{1000}$:



A still open question is: does the sequence $(x_n)_{n\geq 1}$ converge to 0?

"Convergence to 0" means, precisely, for all positive real numbers $\epsilon > 0$ there is a positive integer N such that

$$-\epsilon < x_n < \epsilon$$

for all $n \geq N$.

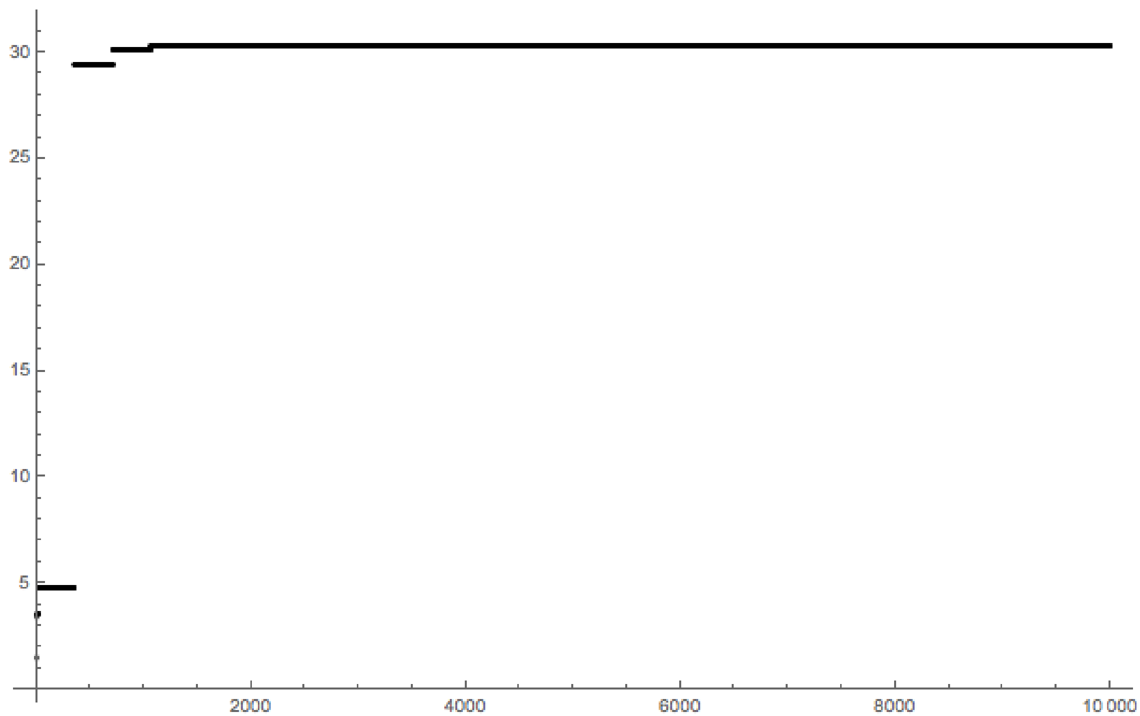
This question is closely related to the irrationality measure of π.

A closely related and still unsolved problem is convergence of the Flint Hills series, which is an infinite series that can be defined in terms of its sequence of partial sums S_n as follows:

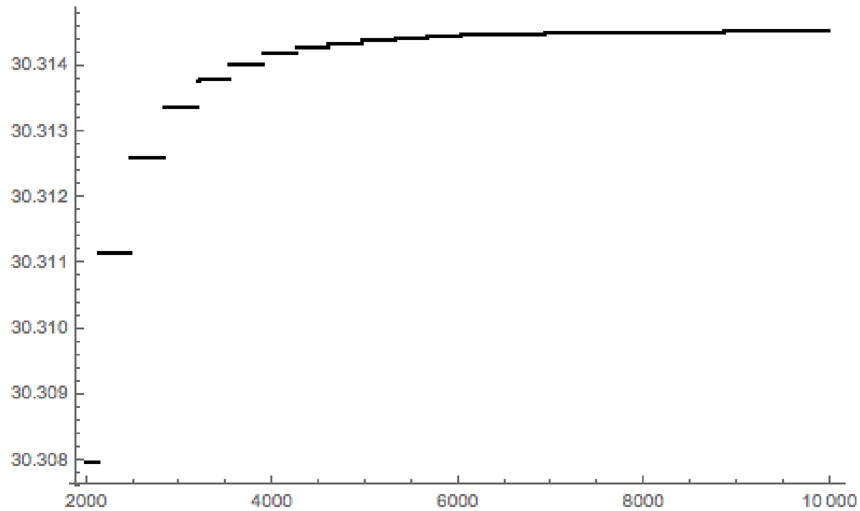
- $S_1 = \frac{1}{\sin(1)}$
- $S_n = S_{n-1} + \frac{1}{n^3 \sin^2(n)}$

The question is whether the sequence of partial sums S_n converges to a limit.

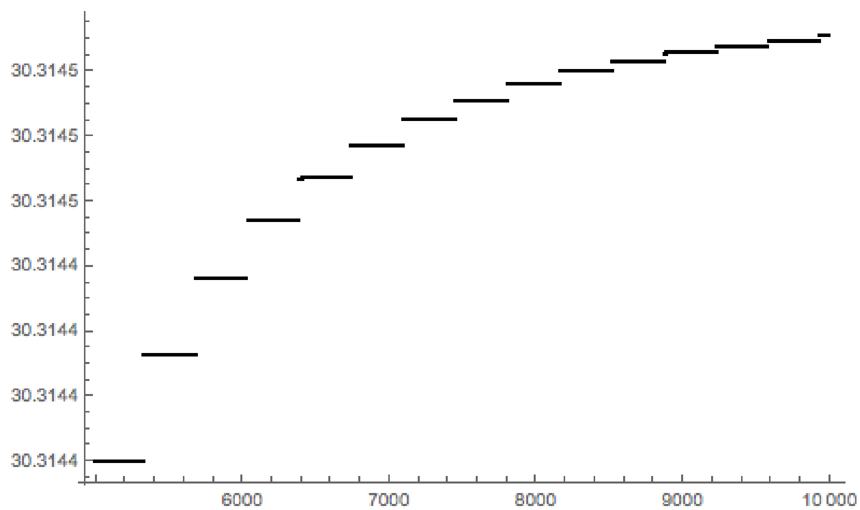
Below is a plot of the first 10,000 partial sums of the Flint Hills series $S_1, S_2, \ldots, S_{10000}$:



The plot below shows the partial sums from S_{2000} through S_{10000}:



while the plot below shows the partial sums from S_{5000} through S_{10000}:



5.14 A lower bound for the size of sin(n)

Computation of the first 10,000,000 values of $\sin(n)$, for n a positive integer, establishes that

$$|\sin(n)| > \frac{1}{4n^2}$$

for $1 \leq n \leq 10^7$.

Is this true for all positive integers n?

If so then

$$\frac{1}{n^2\sqrt{|\sin(n)|}} < \frac{2}{n}$$

for all positive integers n so the sequence

$$\frac{1}{n^2\sqrt{|\sin(n)|}}$$

converges to 0.

Consequently, by a result of Max Alekseyev, the irrationality measure $\mu(\pi)$ of π is no greater than $1 + \frac{2}{1/2} = 5$, which is a substantial improvement on what is known about $\mu(\pi)$.

5.15 Giuga's conjecture on primality

In

> Borwein, D., Borwein, J. M., Borwein, P. B., and Girgensohn, R. (1996). Giuga's conjecture on primality. *The American Mathematical Monthly*, 103(1), 40-50.

the authors discuss a conjecture on primality due to Giuseppe Giuga:

> Giuga, G. (1950). Su una presumibile proprieta caratteristica dei numeri primi. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat.(3), 14(83), 511-528.

who observed that if p is a prime number then

$$s_p := 1 + \sum_{k=1}^{p-1} k^{p-1}$$

is divisible by p

Giuga conjectured this could not happen when p is not prime, so the relevant question is the following:

is there a composite (that is, non-prime) number n for which

$$s_n := 1 + \sum_{k=1}^{n-1} k^{n-1}$$

is divisible by n?